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CAN YOU TAKE SOLOVAY'S INACCESSIBLE AWAY?†

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TABLE OF CONTENTS

§0. Introduction

Solovay, in his celebrated work [7], proves the consistency of "ZF + DC + every set of reals is measurable and has the Baire property". He started from a model with an inaccessible cardinal, so CON(ZF) was not sufficient for his proof. We prove that the inaccessible is necessary for the measurability; in fact, in ZF + DC we can prove:

(a) if there is a set of $\aleph_1$ reals, then there is a non-(Lebesgue)-measurable set of reals.

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(b) If for some real a, $\aleph_1^{L[a]} = \aleph_1$, then some $\mathbf{\Sigma}_3^1$ set of reals is not (Lebesgue) measurable.

On the other hand, we show that for the Baire property the inaccessible is not necessary (i.e., every model ZFC has a generic extension in which every set of reals definable with a real and an ordinal as parameters, has the Baire property). We also show the consistency of the uniformization property (which implies the existence of an embedding; see Remark (2) after 8.1).

We also prove that adding a Cohen real, adding a Souslin tree, and much more, guarantee a completeness theorem for $L_{\omega_1,\omega}$ + Magidor–Malitz quantifier†; however, adding a random real does not guarantee this.††

Lastly, we prove that the $\mathbf{\Sigma}_3^1$ in (a) above is best possible, i.e., every model of ZFC has a generic extension in which every $\mathbf{\Delta}_3^1$-set of reals is Lebesgue measurable. For this we show that iteration of Borel forcing is nice.

For historical comments see Harrington and Shelah [1].

Note that §5 arises from the analysis on how the needed amalgamation of ccc forcing may fail to satisfy the ccc, whereas §7 arises from analyzing why the parallel to §5 fails, i.e., why the amalgamation of two copies of $UM * \underset{\sim}{UM}$ over Cohen generic reals $\underset{\sim}{r}_0$, $\underset{\sim}{r}_1$ satisfy the ccc.

Note that in §1, §2, §4, §5 we construct concrete (and not straightforwardly defined) names, and in §7 we prove our iteration preserves a property (sweetness) without proving that it is preserved by compositions. Those points may have been an obstacle to previous attempts.

Note also that here measure and category are not dual, as usual.

§1. Adding a Souslin tree by Cohen forcing

1.1. THEOREM. *If P is the forcing notion for adding a Cohen real, i.e., P consists of all functions from some n to ω*, then *in V^P there is a Souslin tree.*

PROOF. We construct in V a name for the tree such that, interpreting the name in $V[P]$, we get a Souslin tree. The underlying set of the tree is ω_1, the αth level in the tree is the interval $[\omega \cdot \alpha, \omega \cdot \alpha + \omega) = T_\alpha$. For every $p \in P$ we construct a function $\leqq_p$ from ω_1^2 into $\{\mathbf{t}, \mathbf{f}, \mathbf{i}\}$; intuitively $\leqq_p(\alpha, \beta) = \mathbf{t}$ means

† *Added in fall* 1983. If, in the principles (P) and (P*) (see Definition 2.8), we omit the "Δ-density" requirement the resulting principles $(P)_1$, $(P^*)_1$ remain equivalent (see Lemma 2.9) and are provable in ZFC. Closely related construction principles are lemma 13 (14) of [4] and the simplified morass of Velleman for $\aleph_1$ and [6, §1].

†† Whether it adds a Souslin tree is an open question.

$p \Vdash$ "$\alpha < \beta$," $\leqq_p(\alpha,\beta) = \mathbf{f}$ means $p \Vdash$ "$\alpha \not< \beta$" and $\leqq_p(\alpha,\beta) = \mathbf{i}$ means $p \not\Vdash$ "$\alpha \not< \beta$." Formally, the function $\leqq_p$ will satisfy the following:

(A$_1$) If $\leqq_p(\alpha,\beta) = \mathbf{t}$, $q \geqq p$ then $\leqq_q(\alpha,\beta) = \mathbf{t}$.
If $\leqq_p(\alpha,\beta) = \mathbf{f}$, $q \geqq p$ (q extends p) then $\leqq_q(\alpha,\beta) = \mathbf{f}$.

(A$_2$) If $\leqq_p(\alpha,\beta) = \mathbf{i}$ then there is $q \geqq p$, $\leqq_q(\alpha,\beta) = \mathbf{t}$ and there is $q \geqq p$, $\leqq_q(\alpha,\beta) = \mathbf{f}$.

(A$_3$) For any $p \in P$ and finite $A \subseteq \omega_1 = T$ let $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ be the levels s.t. $A \subseteq \bigcup_{i \leqq n}[\omega \cdot \alpha_i, \omega(\alpha_i + 1))$: Then there is $q \geqq p$ and finite $B \supseteq A$ sitting on the same levels s.t. $\{\langle \alpha,\beta\rangle \mid \leqq_q(\alpha,\beta) = \mathbf{t}\}$ defines a tree on B which is in accordance with the levels, every point not in the last level has at least two extensions in the next level and an extension at any level. Note that this implies that for no $\beta < \alpha + \omega$ and $p \in P$, $\leqq_p(\alpha,\beta) = \mathbf{t}$.

We associate with every $p \in P$ a function $l_p : \omega_1 \to \omega_0$ having the following property:

(B) If $\alpha < \beta$, $l_p(\alpha) = l_p(\beta)$ then $\leqq_p(\alpha,\beta) = \mathbf{i}$ or $\leqq_p(\alpha,\beta) = \mathbf{t}$; and $l_p(\alpha) = l_p(\beta)$ implies α, β are not in the same level or are equal.

First we show how that gives a Souslin tree and then we show how to construct $\leqq_p$, l_p. In $V[G]$ (where G is a generic subset of P over V) define the following partial order: $\alpha \leqq_T \beta \Leftrightarrow$ for some $p \in G$, $\leqq_p(\alpha,\beta) = \mathbf{t}$. We show that it is a tree and then that it is Souslin.

(i) $\leqq_T$ is well founded. Suppose $\alpha_{n+1} <_T \alpha_n$, then we can find $\gamma < \gamma^*$, $n \in \omega$ s.t. $\alpha_n \in T_\gamma$, $\alpha_{n+1} \in T_{\gamma^*}$, and find $p \in G$ s.t. $\leqq_p(\alpha_{n+1}, \alpha_n) = \mathbf{t}$; by (A$_3$) extend p to p' and find a finite tree where $\alpha_{n+1} < \alpha_n$, but then this tree is not in accordance with the levels.

(ii) The other properties of the tree follow from the fact that the elements of G are compatible and from a density argument using (A$_3$). (It might be that limit points in T have the same branch below them, but this does not matter.)

(iii) *The Main point*: T is Souslin — suppose $X \subseteq \omega_1$ is a set of $\aleph_1$ pairwise $<_T$-incompatible elements $r \in G$, $r \Vdash$ "$\underset{\sim}{X}$ is an uncountable set of pairwise $<_T$-incomparable elements". Then find $p \in G$ s.t. $Y = \{\alpha \mid p \Vdash \alpha \in \underset{\sim}{X}\}$ is uncountable, $p \geqq r$ and find $\alpha, \beta \in Y$ s.t. $l_p(\alpha) = l_p(\beta)$; it follows from (B) and (A$_2$) that for some $p' \geqq p$, $\leqq_{p'}(\alpha,\beta) = \mathbf{t}$ so $p' \Vdash \alpha <_T \beta$, a contradiction.

The Construction

We first build the functions $\leqq_p$, l_p on the first ω-levels of the tree — $\bigcup_{n<\omega} T_n$. We will do this by increasing finite approximations ω-many times, at each stage defining the functions for finitely many conditions on a finite subset of $\bigcup_{n\in\omega} T_n$.

1.2 Definitions. *A finite approximation* φ

φ consists of (a) a finite subset A^φ of $\bigcup_{n\in\omega} T_n$, (b) a finite set of conditions $Q^\varphi \subseteq P$ which is closed under initial segments, (c) functions $\leqq_p$, l_p for $p \in Q^\varphi$ defined on A^φ satisfying $(A_{1,2,3})$ and (B) on their domain (for (A_3) restricting ourselves to $A \subseteq A^\varphi$). So if $p \in Q^\varphi$ is maximal in Q^φ then $\leqq_p^\varphi$ determines the partial order on A^φ completely. Let φ^* extend φ, $\varphi \subseteq \varphi^*$ has its natural meaning.

We need the following lemmas to prove that an increasing sequence of finite approximations can be defined on all of $x \in \bigcup_{n\in\omega} T_n$ and P.

1.3. Lemma. *If φ is a finite approximation, $x \in \bigcup_{n\in\omega} T_n$, then there is an extension ψ of φ s.t. $x \in A^\psi$.*

Proof. Easy, by checking.

1.4. Lemma. *If φ is a finite approximation, $p \in P$, then there is an extension φ^p of φ s.t. $p \in Q^\varphi$.*

Proof. By induction on Dom p.

1.5 Definition and Lemma. If φ is a finite approximation, g a finite one-to-one function from a subset of $\bigcup_{n<\omega} T_n$ to $\bigcup_{n<\omega} T_n$, Dom $g \supseteq A^\varphi$, g keeps the levels in their order (i.e., $\{g(x): x \in A^\varphi \cap T_n\} \subseteq T_{h(n)}$, where h is strictly increasing), *then* $\bar\varphi = g(\varphi)$ is defined as follows:

$$Q^{\bar\varphi} = Q^\varphi, \qquad A^{\bar\varphi} = g''A^\varphi$$

$$\text{for } p \in Q^\varphi, \quad \alpha, \beta \in A^\varphi:$$

$$\leqq_p^\varphi(\alpha,\beta) = x \Leftrightarrow \leqq_p^{\bar\varphi}(g(\alpha), g(\beta)) = x \text{ for } x \in \{\mathbf{f}, \mathbf{t}, \mathbf{i}\},$$

$$l_p^\varphi(\alpha) = l_p^{\bar\varphi}(f(\alpha)).$$

Then $\bar\varphi$ is a finite approximation.

1.6. Duplication Lemma. *If φ is a finite approximation, $A^\varphi \subseteq \bigcup_{l<n} T_l$, $k < n$, g is appropriate for φ (i.e., satisfying the definition and lemma above), $g \restriction \bigcup_{l<k} T_l$ is the identity and, $\forall k \leqq l < n$, $g: T_l \to T_{n+l}$* then *there is a finite approximation ψ extending φ and $g(\varphi)$.*

So let us define ψ; we shall later define Q^ψ such that $Q^\varphi \cup Q^{g(\varphi)} \subseteq Q^\psi$, we define $A^\psi = A^\varphi \cup A^{g(\varphi)}$. For α, $\beta \in A^\psi$ and $p \in Q^\varphi$ we define $\leqq_p^\psi(\alpha,\beta)$ by cases:

Case I: *p maximal*

(a) $\leqq^{\psi}_{p}(\alpha,\beta)=\leqq^{\varphi}_{p}(\alpha,\beta)$ if $\alpha,\ \beta\in A^{\varphi}$;

(b) $\leqq^{\psi}_{p}(\alpha,\beta)=\leqq^{g(\varphi)}_{p}(\alpha,\beta)$ if $\alpha,\ \beta\in A^{g(\varphi)}$;

(c) $\leqq^{\psi}_{p}(\alpha,\beta)=\mathbf{i}$ if $\alpha+\omega\leqq\beta$ and for every $\gamma\in A^{\varphi}\cap A^{g(\varphi)}$, $\leqq^{p}_{\varphi}(\gamma,\alpha)=\mathbf{t}$ iff $\leqq^{p}_{\varphi}(\gamma,\beta)=\mathbf{t}$;

(d) $\leqq^{\psi}_{p}(\alpha,\beta)=\mathbf{f}$ otherwise.

Case II: *p not maximal*

For $x\in\{\mathbf{t},\mathbf{f}\}$, $\leqq^{\psi}_{p}(\alpha,\beta)=x$ iff for every maximal $r\in Q^{\varphi}$,

$$r\geqq p\Rightarrow\leqq^{\psi}_{r}(\alpha,\beta)=x;\qquad\text{otherwise}\quad\leqq^{\psi}_{p}(\alpha,\beta)=\mathbf{i}.$$

Lastly, for $p\in Q$ we let $l^{\psi}_{p}=l^{\varphi}_{p}\cup l^{g(\varphi)}_{p}$.

It is easy to check that (A_1), (B) hold (when $p,\ q\in Q^{\varphi}$, $p\in Q^{\varphi}$ resp.), however (A_2), (A_3) do not necessarily hold. So for each maximal member p of Q^{φ}, $l<2$ and $\alpha,\ \beta\in A^{\psi}$ we shall choose an immediate successor $q^{l}_{p,\alpha,\beta}$ of p (in P) such that $q^{l}_{p,\alpha,\beta}\notin Q^{\varphi}$ and $q^{l^1}_{p_1,\alpha_1,\beta_1}=q^{l^2}_{p_2,\alpha_2,\beta_2}$ implies $p_1=p_2$, $\alpha_1=\alpha_2$, $\beta_1=\beta_2$, $l^1=l^2$. Now we let

$$Q^{\psi}=Q^{\varphi}\cup\{q^{l}_{p,\alpha,\beta}:p\in Q^{\varphi}\text{ is maximal in }Q^{\varphi}\text{ and }l=0,1\}.$$

Then we define, for $q=q^{l}_{p,\alpha,\beta}\in Q^{\psi}-Q^{\varphi}$, l^{ψ}_{q} as any one-to-one function from A^{ψ} into ω. Lastly we shall define for any such q, $\leqq^{\psi}_{q}$; it determines a tree on A^{ψ}, $[\leqq^{\psi}_{p}(\alpha_1,\beta_1)\in\{\mathbf{t},\mathbf{f}\}\Rightarrow\leqq^{\psi}_{p}(\alpha_1,\beta_1)=\leqq^{\psi}_{q}(\alpha_1,\beta_1)]$ and $\leqq^{\psi}_{q}(\alpha,\beta)=\mathbf{t}$ if $l=0$, $\leqq^{\psi}_{p}(\alpha,\beta)\neq\mathbf{f}$ and $\leqq^{\psi}_{q}(\alpha,\beta)=\mathbf{f}$ if $l=1$, $\leqq^{\psi}_{q}(\alpha,\beta)\neq\mathbf{t}$.

Why can this be done? If $\leqq^{\psi}_{p}(\alpha,\beta)=\mathbf{i}$, $l=0$, by the definition of $\leqq^{\psi}_{p}$, necessarily for some γ in the last level of $A^{\varphi}\cap A^{g(\varphi)}$, $\leqq^{\varphi}_{p}(\gamma,\alpha)=\mathbf{t}$. $\leqq^{g(\varphi)}_{p}(\gamma,\beta)=\mathbf{t}$ so we can easily complete the tree as required. If $\leqq^{\psi}_{p}(\alpha,\beta)=\mathbf{t}$, $l=1$ our tree is easier as well as in the other cases.

Now ψ is as required. Condition (B) holds. For $p\in Q^{\psi}-Q^{\varphi}$, l^{ψ}_{p} is one-to-one. Suppose $\leqq^{\psi}_{p}(\alpha,\beta)=\mathbf{f}$, $\alpha\leqq\beta$, $p\in Q^{\varphi}$, $l^{\psi}_{p}(\alpha)=l^{\psi}_{p}(\beta)$, then the only non-trivial case is $\alpha\in A-A^{g(\varphi)}$, $\beta\in A^{g(\varphi)}-A^{\varphi}$; then $l_p(\alpha)=l_p(g^{-1}(\beta))$. If $\alpha=g^{-1}(\beta)$ case (c) above has to occur. Otherwise necessarily α, $g^{-1}(\beta)$ are not in the same level, and $\leq^{\varphi}_{p}(\alpha,g^{-1}(\beta))\neq\mathbf{f}$ or $\leqq^{\varphi}_{p}(g^{-1}(\beta),\alpha)\neq\mathbf{f}$ (as p satisfies condition (B)), hence for some q, maximal in A^{φ}, $p\leqq q\in Q^{\varphi}$, $\leqq^{\varphi}_{q}(\alpha,g^{-1}(\beta))=\mathbf{t}$ or $\leqq^{\varphi}_{q}(g^{-1}(\beta),\alpha)=\mathbf{t}$, but this contradicts $\leqq^{\psi}_{p}(\alpha,\beta)=\mathbf{f}$ by (c) above.

Condition (A_1) is immediate, and (A_3) holds by the choice of the $q^{l}_{p,\alpha,\beta}$. As for (A_2) let $p\in Q^{\psi}$, $\leqq^{\psi}_{p}(\alpha,\beta)=\mathbf{i}$, $x\in\{\mathbf{t},\mathbf{f}\}$, necessarily $p\in Q^{\varphi}$. If $\alpha,\ \beta\in A^{\varphi}$ for some r, $p\leqq r\in Q^{\varphi}$, $\leqq^{\varphi}_{r}(\alpha,\beta)=x$. So r is as required. If $\alpha,\ \beta\in A^{g(\varphi)}$ the proof is similar. So suppose $\alpha\in A^{\varphi}-A^{g(\varphi)}$, $\beta\in A^{g(\varphi)}-A^{\varphi}$, and $x=\mathbf{t}$. By the definition of $\leqq_p^\psi$, necessarily some maximal r, $p \leqq r \in Q^\varphi$, is as required in (c). Clearly $\leqq_r^\psi(\alpha, \beta) = \mathbf{i}$. Now $q^0_{r,\alpha,\beta}$ will be as required. If $x = \mathbf{f}$ the proof is easier.

1.7 Definition. Let $\delta \in \omega_1$ be limit. Suppose for $p \in P$ we have defined $\leqq_p$, l_p on $T \restriction \delta = \bigcup_{i<\delta} T_i$ satisfying $(A_{1,2,3})$, (B); we say

$$\Phi = \{\leqq_p \restriction (T \restriction \delta), l_p \restriction (T \restriction \delta) : p \in P\}$$

has the *duplication property* if:

(1) For any finite $Q \subseteq P$ and $A \subseteq T \restriction \delta$ there is a finite approximation $\varphi \subseteq \Phi$ s.t. $A^\varphi \supseteq A$, $Q^\varphi \supseteq Q$. (We are redefining finite approximation allowing $A^\varphi \subseteq \omega_1$.)

(2) For any finite approximation $\varphi \subseteq \Phi$, $A^\varphi \subseteq \bigcup_{l \leq n} T_{\alpha_l}$, $k \leqq n$ and $\alpha_n < \gamma < \delta$, there is f, $\operatorname{Dom} f \supseteq A^\varphi$, f preserves the levels (is appropriate) s.t. $f \restriction \bigcup_{l<k} T_{\alpha_l} =$ identity, $f''T_{\alpha_l} \subseteq (T \restriction \delta) - (T \restriction \gamma)$ for $k \leqq l \leqq n$ such that $f(\varphi) \subseteq \Phi$.

Using the previous lemmas we can find Φ_ω with the duplication property on $T \restriction \omega$. We want to extend this on all of $T \restriction \omega_1$. (There is a general theorem for construction. See [4], [5], lemmas 13, 14.)

1.8. Fact. If Φ_δ ($\delta < \alpha$, α limit) is an increasing sequence of Φ_δ's having the duplication property, then $\bigcup_{\delta<\alpha} \Phi_\delta = \Phi_\alpha$ has the duplication property for $T \restriction \alpha$.

1.9. Lemma. *If Φ_δ has the duplication property, then there is $\Phi_{\delta+\omega} \supseteq \Phi_\delta$ with the duplication property on $T \restriction (\delta + \omega)$.*

Proof. We will get $\Phi_{\delta+\omega}$ as an increasing ω-sequence of finite approximation φ s.t. $\varphi \restriction (T \restriction \delta) \subseteq \Phi_\delta$ and, moreover, if $T_{i_1}, \ldots, T_{i_n}$ are the levels of $A^\varphi \cap (T \restriction \delta)$, then there is a function f, s.t. $f \restriction (T_{i_1} \cup \cdots \cup T_{i_n})$ is the identity and $f(\psi) = \varphi$ for some finite approximation $\psi \subseteq \Phi_\delta$ s.t. $\psi \restriction (T \restriction (i_n + 1)) = \varphi \restriction (T \restriction \delta)$. We say that such φ has source, in Φ_δ, and it is easy to extend such φ's.

§2. The principle (P), Magidor–Malitz quantifiers and adding a Cohen real

2.1. Notation. $S_\lambda(A) = \{B : B \subseteq A, |B| < \lambda\}$; $\mathrm{Lim} = \{\delta < \omega_1 : \delta$ a limit ordinal$\}$.

Let s, t denote finite subsets of ω_1; $s \trianglelefteq t$ (s an initial segment of t) if $\alpha \in s$, $\beta < \alpha$, $\beta \in t$ implies $\beta \in s$;

$$s \lhd t \quad \text{if } s \trianglelefteq t, \qquad s \neq t.$$

Let $s < t$ mean $(\forall \alpha \in s)(\forall \beta \in t)(\alpha < \beta)$.

$\mathrm{Val}(s, t)$ is the following function f: if $s = \{\alpha_0, \ldots, \alpha_{n-1}\}$, $t = \{\beta_0, \ldots, \beta_{n-1}\}$ (increasing), then

$$f(l,k)=0 \quad \text{iff } \alpha_l=\beta_k,$$

$$f(l,k)=1 \quad \text{iff } \alpha_l<\beta_k,$$

$$f(l,k)=2 \quad \text{iff } \alpha_l>\beta_k.$$

2.2. DEFINITION. A partial order P is a ccc-indiscernible forcing if

(A) $P=\{\tau_{n,m}(t): n,m<\omega,\ t\subseteq\omega_1,\ |t|=n\}$; for $p\in P$, $\operatorname{Dom} p=t$ if $p=\tau_{n,m}(t)$, and σ,τ range over $\{\tau_{n,m}: n,m<\omega\}$; by writing $\tau(s)$ we mean $|s|$ is appropriate.

(B) *Indiscernibility.* The truth value of $\tau(s)<\sigma(t)$ (in P) depends only on τ,σ and $\operatorname{Val}(s,t)$ (hence $|s|,|t|$ too).

(C) P satisfies the ccc (countable chain condition), equivalently for $s<t_1<t_2$, $|t_1|=|t_2|$, $\tau(s\cup t_1)$, $\tau(s\cup t_2)$ are compatible.

2.3. DEFINITION. We say P is a smooth ccc-indiscernible forcing if in addition:

(D) If p,q are compatible in P then they have an upper bound r, $\operatorname{Dom} r=\operatorname{Dom} p\cup\operatorname{Dom} q$.

(E) If $p\in P$, $t=\operatorname{Dom} p$, $s\lhd t$ then there is $q=p\restriction s$ such that

  (i) $q\leqq p$, and

  (ii) $q\leqq r$, $(\operatorname{Dom} r)\cap t=s$, implies p and r are compatible.

(F) $\tau(s)\leqq\sigma(t)$ implies $s\subseteq t$.

REMARK. Note that (E) implies (C).

2.4. DEFINITIONS.

(1) D is an indiscernibility density function, if for any finite $s\subseteq\omega_1$, $D(s)$ is a dense subset of P, and the truth value of $\tau(t)\in D(s)$ depends only on τ and $\operatorname{Val}(s,t)$.

(2) D is smooth if for every s and $p\in P$ there are $r,q\in P$, $p\leqq r$, $q\leqq r$, $q\in D(s)$ and $\operatorname{Dom} q=s$.

2.5. DEFINITION. We call E a Δ-density set if

(A) for some $n=n(E)$, it is a set whose members have the form $\langle\tau(s\cup t_1),\dots,\tau(s\cup t_n),\tau^*(t^*)\rangle$, $s<t_1<t_2<\cdots<t_n$, $\tau(s\cup t_l)\leqq\tau^*(t^*)$;

(B) the indiscernibility condition is satisfied;

(C) if $s\subseteq s'$, $t_e\subseteq t'_e$, $s\lhd s\cup t_e$, $s\cap t_e=\emptyset$, $s'\lhd s'\cup t'_e$, $s'\cap t'_e=\emptyset$, $\operatorname{Val}(t_e,t'_e)=\operatorname{Val}(t_1,t'_1)$, $\tau(s\cup t_e)\leqq\sigma(s'\cup t'_e)$, $s<t_1<t_2<\cdots$, $s'<t'_1<\cdots$ and $\operatorname{Min} t_e$ is limit, then there are $\sigma^*(t)$, $\tau^*(t^*)$ such that

$$\langle\tau(s\cup t_1),\dots,\tau(s\cup t_n),\tau^*(t^*)\rangle\in E,$$

and $\sigma(s' \cup t'_e) \leqq \sigma^*(t)$ and $\tau^*(t^*) \leqq \sigma^*(t)$.

2.6. DEFINITION. We call a Δ-density set smooth if in (A) $t^* = s \cup \bigcup_e t_e$, and in (C) $t^* = s' \cup \bigcup_e t'_e$ (so we can omit "Min t_e is limit").

2.7. DEFINITION. Suppose P is a ccc- indiscernible forcing, $G \subseteq P$ is directed.

(1) G satisfies a density function D if for any $s \in S_{\aleph_0}(\omega_1)$, $G \cap D(s) \neq \emptyset$.

(2) G satisfies a Δ-density set E *if* for any $\tau(s \cup t_i) \in G$ $(i < \omega_1)$ such that $s < t_i < t_j$ for $i < j$, $|t_i| = |t_0|$ *there are* $i(1) < \cdots < i(n)$, $n = n(E)$, and $\tau^*(t^*) \in G$ such that $\langle \tau(s \cup t_{i(1)}), \ldots, \tau(s \cup t_{i(n)}), \tau^*(t^*) \rangle \in E$.

2.8. DEFINITIONS.

(1) The principle (P) holds iff for any ccc-indiscernible forcing P, and countably many density functions and Δ-density sets, there is a directed $G \subseteq P$ satisfying them.

(2) The principle (P^s) holds iff for any smooth ccc-indiscernible forcing P, and countably many smooth density functions and smooth Δ-density sets, there is a directed $G \subseteq P$ satisfying them.

(3) We define the principles $(\mathrm{P})_n$, $(\mathrm{P}^s)_n$ similarly, but restricting ourselves to Δ-density sets E, $n(E) \leqq n$.

2.9 LEMMA.

(1) *The principles* (P) *and* (P^s) *are equivalent.*

(2) *Also,* $(\mathrm{P})_n$ *and* $(\mathrm{P}^s)_n$ *are equivalent.*

PROOF. Trivially the principle (P) implies the principle (P^s). So suppose the principle (P^s) holds, P is a ccc-indiscernible forcing, D_l $(l < \omega)$ are density functions, and E_l $(l < \omega)$ Δ-density sets for it. Let $T_0 = \{\tau_{n,m} : n < \omega\}$ and $S = \{\delta : \delta < \omega_1$ limit ordinal$\}$.

First we can assume that in P, for every $\tau_1(s), \tau_2(t) \in P$ which are compatible, there is a least upper bound $\sigma(s \cup t) = \tau_1(s) \cup \tau_2(t)$ and $\tau(s) \leqq \sigma(t)$ implies $s \subseteq t$.

For this, let $P' = \{\langle s, \Gamma \rangle : s \in S_{\aleph_0}(\omega_1)$, Γ a finite subset of P such that $\sigma(t) \in P \Rightarrow t \subseteq s$ and Γ has an upper bound$\}$.

On P' we define an order: $\langle s_1, \Gamma_1 \rangle \leqq \langle s_2, \Gamma_2 \rangle$ if $s_1 \subseteq s_2$, and $\Gamma_1 \subseteq \Gamma_2$. It is easy to check that P is a ccc forcing, satisfying the above-mentioned conditions, and it suffices to prove the principle (P) for it (i.e., we can translate the problem of finding directed $G \subseteq P$ to the problem of finding $G' \subseteq P'$). Note

2.10. FACT. If $p \in P$, $\delta < \omega_1$ is limit, then there is $q \in P$, $\mathrm{Dom}\, q \subseteq \delta$, such that for every q', if $q \leqq q' \in P$, $\mathrm{Dom}\, q' \subseteq \delta$ then q' is compatible with p.

PROOF. Suppose not; let $p = \tau(s \cup t)$ where $s = (\operatorname{Dom} p) \cap \delta$ and $t = \operatorname{Dom} p - s$. Now choose t_i $(i < \omega_1)$ such that $t_i \subseteq [\delta + i\omega, \delta + i\omega + \omega)$ and $|t_i| = t$. By the indiscernibility, and as we have assumed that p, δ form a counterexample, for each $i < \omega_1$ there is $q_i \in P$ such that $\tau(s \cup t_i) \leqq q_i$, $\operatorname{Dom} q_i \subseteq \delta + i\omega + \omega$ and q_i is incompatible with $\tau(s \cup t_{i+1})$. Clearly, by the indiscernibility for $i < j$, q_i is incompatible with $\tau(s \cup t_j)$; but $\tau(s \cup t_i) \leqq q_i$, so the q_i's are $\aleph_1$ pairwise incompatible members of P, a contradiction.

So for every p, δ there is a condition q, $\operatorname{Dom} q \subseteq \delta$, such that $r = q \cup p$ satisfies: $q \leqq r$, $\operatorname{Dom} q \subseteq \delta$, and they satisfy: $q' \geqq q$, $\operatorname{Dom} q' \subseteq \delta$ implies q', r are compatible. For any $r \in P$, let δ_0 be maximal $\delta \in \mathrm{Lim}$ such that $[\delta, \omega_1] - \operatorname{Dom} r \neq \emptyset$, and let $r_1 = q_0 \cup r$ be as above. Now let $\delta_1 < \delta_0$ be maximal such that $[\delta_1, \delta_0] - \operatorname{Dom} q_1 \neq \emptyset$ and find suitable $r_2 = q_1 \cup r_1$, $\operatorname{Dom} q_1 \subseteq \delta_1$, and continue. Eventually, as the ordinals are well ordered, we find $r^* \geqq r$ such that for every limit ordinal, there is $r^* \restriction \delta$ such that $q' \geqq r^* \restriction \delta$, $\operatorname{Dom} q' \subseteq \delta$ implies q', r^* are compatible. (Why not only for $\delta \in \{\delta_0, \delta_1, \ldots, \}$? By the indiscernibility.) Call the set of such r^*, P^*. P^* does not satisfy the indiscernibility condition (even $\tau(t) \in P^*$ does not), but when we replace ω_1 by Lim, adding more terms, it will, and it is a smooth ccc indiscernible forcing.

Now we have to "translate" the D_l's and E_l's and we are done.

It is quite obvious (see Magidor–Malitz [3]) that

2.11. THEOREM. *The principle* (P) *is equivalent to the completeness theorem for* $L_{\omega_1,\omega}(Q_1^{\mathrm{MM}}, Q_2^{\mathrm{MM}}, \ldots, Q_n^{\mathrm{MM}}, \ldots)$. *Similarly for* $(\mathrm{P})_n$, $L_{\omega_1,\omega}(Q_n^{\mathrm{MM}})$.

See [6] for how easy it is to use (an older variant of) (P) for many applications.

There are two differences between (P) and the principle from [6].

(a) More of the work of applying the principle is put into the principle there, so it has less of the flavour of a combinatorial principle.

(b) The principle there is a little stronger.

But we could have used various variants of the two and everything is parallel (except 2.11, where the logic has to be changed). Note

2.12. CONCLUSION. The principle (P^s) follows from $\diamondsuit_{\aleph_1}$.

2.13. THEOREM. *If we add to a universe of set theory a Cohen generic real, then the resulting model satisfies the principle* (P^s).

PROOF. We can use as Cohen forcing ${}^{\omega>}\omega$ with the order $\lhd$ (being initial segment); we call this forcing notion Q. So we are given Q-names $\underset{\sim}{P}$, $\underset{\sim}{D}_n$, $\underset{\sim}{E}_n$ $(n < \omega)$, which are forced to be as in Definition 2.8(2), and we have to construct a Q-name $\underset{\sim}{G}^P$ of a directed subset of P satisfying $\underset{\sim}{D}_n$, $\underset{\sim}{E}_n$; the problematic part is satisfying $\underset{\sim}{E}_n$.

Note that we know the set of elements of P. For seeing what we need to satisfy $\underset{\sim}{E}_n$, let $\underset{\sim}{G}^P$ be a Q-name of a directed subset of P.

So suppose $q \in Q$ forces that $\underset{\sim}{E} = \underset{\sim}{E}_l$, $\underset{\sim}{\tau}(\underset{\sim}{s} \cap \underset{\sim}{t}_i)$ $(i < \omega_1)$ form a counterexample. For each i there is $q_i \geqq q$, $q_i \Vdash$ "$\underset{\sim}{\tau} = \tau_i$, $\underset{\sim}{s} = s_i$, $n(\underset{\sim}{E}) = n_i$, $\underset{\sim}{t}_i = t_i$". But there are only countably many $q_i \in {}^{\omega>}\omega$, hence for some q^*, $q_i = q^*$ for $i \in S$, $S \subseteq \omega_1$ uncountable. So $q^* \Vdash$ "$\underset{\sim}{s} = s^*$ and $n(\underset{\sim}{E}) = n^*$" for some s^*, n^*; and it suffices to guarantee the existence of $q^+ \geqq q^*$ and n, $i(1) < \cdots < i(n)$ in S, and

$$q^+ \Vdash_Q \text{“}\langle \tau(s, t_{i(1)}), \ldots, \tau(s, t_{i(n)}), \tau^*(t^*)\rangle \in \underset{\sim}{E} \quad \text{and } \tau^*(E^*) \in \underset{\sim}{G}^P\text{”}.$$

From now till the end of the proof of Theorem 2.13:

2.14. Assumption. $\underset{\sim}{P}$, $\underset{\sim}{D}_l$, $\underset{\sim}{E}_e$ are Q-names as in Definitions 2.3, 2.4(2) and 2.6 for l, $e < \omega$.

2.15. Definition. We shall define here what is a finite system. The finite systems are approximations to a full system, which gives a name $\underset{\sim}{G}^P$ as required.

A finite system S is consistent with the following (when several systems are discussed, S will have an additional superscript, or we use $S(1)$, etc.):

(A) *The domain*: A finite subset W of ω_1 and let $n(*) = |W|$, and a finite subset Γ of ${}^{\omega>}\omega$ closed under initial segments, and

(B) *The "forcing relation"*: A function G, with domain Γ such that for $\eta \in \operatorname{Dom} G$, $G(\eta)$ is a finite set of elements of the form $\tau(s)$, $s \subseteq W$ and $\eta \Vdash_Q$ "in P, the set $G(\eta)$ has an upper bound".

(C) *The local ignorance condition*: For every $\eta \in \Gamma$ there is $k(\eta) < \omega$ and there is a family $H(\eta)$ of subsets of W such that
  - (i) $A \in H(\eta)$ implies $|A| = k(\eta)$,
  - (ii) if $A, B \in H(\eta)$ then $A \cap B$ is an initial segment of A (and of B),
  - (iii) if $\tau(t) \in G(\eta)$ then, for some $A \in H(\eta)$, $t \subseteq A$,
  - (iv) if A, $B \in H(\eta)$, h the unique order preserving function from A onto B, and $\alpha_1, \ldots \in A$, then

$$\tau(\{\alpha_1, \alpha_2, \ldots\}) \in G(\eta) \quad \text{iff } \tau(\{h(\alpha_1), h(\alpha_2), \ldots\}) \in G(\eta).$$

Now a system consists of W, Γ, G, H, provided they are as mentioned above and satisfy:

(D) *Monotonicity of G, H*: If $\nu \lhd \eta$ are in Γ then $G(\nu) \subseteq G(\eta)$, and for every $A \in H(\nu)$ there is $B \in H(\eta)$, $A \subseteq B$.

2.16. Subclaim. In Definition 2.15(B), instead of

$$\eta \Vdash_Q \text{“in } P\text{, the set } G(\eta) \text{ has an upper bound”}$$

it suffices to demand

$$\eta \Vdash_Q \text{“in } P\text{, the set } G_A(\eta) \text{ has an upper bound for any (some) } A \in H(\eta)\text{”}$$

where

$$G_A(\eta) = \{\tau(t) \in G(\eta) : t \subseteq A\}.$$

Note that $G(\eta) = \bigcup \{G_A(\eta) : A \in H(\eta)\}$.

Proof. Easy by property (E) from Definition 2.3: We can define, by induction on $\alpha \in \bigcup_{p \in G(\eta)} \operatorname{Dom} p \cup \{\omega_1\}$, an upper bound for $\{p \restriction \alpha : p \in G(\eta)\}$.

2.17. Definition. A partial order $<$ is defined on the set of finite aproximations,

$S(1) < S(2)$ *if*: $W^{S(1)} \subseteq W^{S(2)}$, $\Gamma^{S(1)} \subseteq \Gamma^{S(2)}$, for every $\eta \in \Gamma^{S(1)}$ $H^{S(1)}(\eta) \subseteq H^{S(2)}(\eta)$, for $A \in H^{S(1)}(\eta)$ $G_A^{S(1)}(\eta) = G_A^{S(2)}(\eta)$ (note the equality), hence $G^{S(1)}(\eta) \subseteq G^{S(2)}(\eta)$.

2.18. Subclaim. The set AP of finite approximations is partially ordered by $<$, and it has the countable chain condition.

Proof. Trivial, if we use Subclaim 2.16.

2.19. Claim. Suppose $S \in AP$, $\eta \in \Gamma^S$, then there are $S(1)$, $S < S(1)$, and ν such that:

(i) $\eta \lhd \nu$, $\nu \in \Gamma^{S(1)}$,

(ii) $H^S(\eta) \subseteq H^{S(1)}\nu$,

(iii) for any $t \in H^{S(1)}(\nu)$ there is $\tau(t) \in G_t^{S(1)}(\nu)$ and

$$\nu \Vdash_Q \text{“}\tau(t) \text{ is an upper bound of } G_t^{S(1)}(\nu)\text{”}.$$

(When (iii) holds we say ν is canonical in $S(1)$, for $\tau(t)$. Hence it is canonical in any $S(2) > S(1)$. If we omit "for $\tau(t)$" it means "for some $\tau(t)$".)

Proof. Easy.

2.20. Claim. Suppose $S \in AP$, $\eta \in \Gamma^S$ is canonical in S for $\tau(t)$, and $\eta \Vdash \text{“}n(\underset{\sim}{E}_l) = n\text{”}$, $s \cup t_1, \dots, s \cup t_n \in H^S(\eta)$, $s < t_1 < \cdots < t_n$, $|t_1| = |t_2| = \cdots = |t_n|$, $t'_m \subseteq t_m$, $\operatorname{Val}(t'_m, t_m) = \operatorname{Val}(t'_1, t_1)$, $s' \subseteq s$, $\tau'(s' \cup t'_m) \leqq \tau(s \cup t_m)$.

Then there are $S(1) \in AP$, $S(1) > S$, and $\nu \in \Gamma^{S(1)}$, $\eta \lhd \nu$ and $\tau^*(t^*)$ such that

$$\nu \Vdash_Q \text{“}\langle \tau'(s' \cup t_1'), \ldots, \tau'(s \cup t_n'), \tau^*(t^*)\rangle \in \underset{\sim}{E}_l\text{”}$$

and $\tau^*(t^*) \in G^{S(1)}(\nu)$.

PROOF. Let $t^* = s \cup \bigcup_{l=1} t_l$.

Choose m such that $\eta\,\hat{}\,\langle m \rangle \notin \Gamma^S$, and choose ν, $\eta\,\hat{}\,\langle m \rangle \lhd \nu \in {}^{\omega>}\omega$, $\nu \Vdash_Q$ “$\langle \tau'(s \cup t_1), \ldots, \tau'(s \cup t_n), \tau^*(t^*)\rangle \in E_l$ such that $\{\tau^*(t^*)\} \cup \{\tau(s, t_l) : l = 1, n\}$ has an upper bound” for some $\tau^*(t^*)$. This is possible by 2.5(C). Now the only problem is to show $\nu \Vdash_Q$ “$G(\eta) \cup \{\tau^*(t^*)\}$ is compatible”. This is quite easy by Definition 2.3(E). (So we let $\Gamma^{S(2)} = \Gamma^S \cup \{\nu \restriction l : l \leqq l(\nu)\}$ for $l(\eta) \leqq l < l(\nu)$, $G^{S(1)}(\nu \restriction l) = G^S(\eta)$, $G^{S(2)}(\nu) = \{\tau'(t')\}$ where $\tau'(t')$ is an upper bound of $G(\eta) \cup \{\tau^*(t^*)\}$, $t' = W^S$.)

2.21. CLAIM. Suppose D is a smooth density function. If $S \in AP$, $\eta \in \Gamma^S$, $l < \omega$, $t \subseteq W^S$ then there is $S(1) > S$, $\nu \in \Gamma^{S(1)}$, $\eta \lhd \nu$, and $\tau(t) \in G^{S(1)}(\nu)$, $\nu \Vdash_Q$ “$\tau(t) \in \underset{\sim}{D}_l(t)$”.

PROOF. Trivial.

2.22. LEMMA. *There is $L \subseteq AP$ such that:*

(1) *L is directed.*

(2) *For any $\alpha < \omega_1$ for some $S \in L$, $\alpha \in W^S$.*

(3) *For any $\eta \in {}^{\omega>}\omega$ for some $S \in L$, $\eta \in \Gamma^S$.*

(4) *For any $\eta \in {}^{\omega>}\omega$, $l < \omega$, $t \in S_{\aleph_0}(\omega_1)$ there are $S \in L$, $\nu \in \Gamma^S$, $\eta \lhd \nu$, and $\tau(t) \in G^S(\nu)$ s.t. $\nu \Vdash_Q$ “$\tau(t) \in \underset{\sim}{D}_l(t)$”.*

(5) *For any $\eta \in {}^{\omega>}\omega$, $l < \omega$, $S \in L$, there are $\nu \in {}^{\omega>}\omega$, $\eta \lhd \nu$, $S(1) \in L$, $S < S(1)$ such that $\nu \in \Gamma^{S(1)}$ is canonical in $S(1)$ and $\nu \Vdash_Q$ “$n(\underset{\sim}{E}_l) = n$” for some n.*

(6) *Suppose $\eta \in {}^{\omega>}\omega$ is canonical for $S \in L$, $s \cup t_1, \ldots, s \cup t_n \in H^S(\eta)$, $\eta \Vdash$ “$n(\underset{\sim}{E}_l) = n$”, $s < t_1 < \cdots < t_n$, $s' \subseteq s$, $t_m' \subseteq t_m$ and*

$$\mathrm{Val}(t_m', t_m) = \mathrm{Val}(t_1', t_1), \qquad \tau'(s' \cup t_m') \leqq \tau(s \cup t_m).$$

Then there are $S(1) \in L$, $S(1) > S$, $\nu \in {}^{\omega>}\omega$, $\eta \lhd \nu$, $\nu \in \Gamma^{S(1)}$, $\tau^*(t^*)$ such that

$$\nu \Vdash_Q \text{“}\langle \tau'(s \cup t_1'), \ldots, \tau'(s' \cup t_n'), \tau^*(t^*)\rangle \in \underset{\sim}{E}_l\text{”}, \qquad \tau^*(t^*) \in G^{S(1)}(\nu).$$

PROOF. The existence of such S is equivalent to the existence of a model of some $\psi \in L_{\omega_1,\omega}(Q)$ (Q — the quantifier “there are uncountably many x such that...”). By Keisler [2][†] this is absolute (as long as $\aleph_1$ remains $\aleph_1$), so it suffices

[†] Namely, the completeness theorem for $L_{\omega_1,\omega}(Q)$.

to find a generic extension of our universe in which $\aleph_1$ is not collapsed, and such L exists there. Use AP as a forcing notion; by 2.18 this does not collapse $\aleph_1$, and use the generic subset of AP as the desired L. It satisfies (1) trivially, (2), (3) easily, (4) by Claim 2.21, (5) by Claim 2.19, and (6) by Claim 2.20.

PROOF OF THEOREM 2.13. Let L be as in Lemma 2.22 and define the Q-name $\underset{\sim}{G}^P$:

$$\underset{\sim}{G}^P = \{\tau(t)\text{: for some } \eta \in {}^{\omega>}\omega\text{, which is in the generic subset of } Q\text{, and } S \in L,\ p \in G^S(\eta)\}.$$

It is easy to check $\underset{\sim}{G}^P$ is as required.

§3. Adding a random real is different

3.1. THEOREM. *If V is a universe of set theory, r a random real over V, then in $V[r]$, $(\mathrm{P})_3$ does not necessarily hold. In fact, not necessarily, there are h, $<^*$ such that:*

(a) *$(\omega_1, <^*)$ is a tree, the last level is $\omega_1 - \omega$; it is the ω-th level, so that for every $\alpha \in [\omega, \omega_1]$, $\{l : l <^* \alpha\} \subseteq \omega$ has order type ω (by $<^*$),*

(b) *for any $\alpha \neq \beta \in [\omega, \omega_1]$, $h(\alpha,\beta) <^* \alpha$, $h(\alpha,\beta) <^* \beta$ and there is no bigger element (by $<^*$) with those properties,*

(c) *for any uncountable $S \subseteq \omega_1$ there are (distinct) α, β, $\gamma \in S$, s.t. $h(\alpha,\beta) = h(\alpha,\gamma) = h(\beta,\gamma)$.*

REMARK. *If $V = L$, then in $V[r]$ obviously $\diamondsuit_{\aleph_1}$ holds; and even* CH *implies the existence of such a tree.*

PROOF. It will be enough to assume V satisfies Martin's Axiom and $2^{\aleph_0} > \aleph_1$. Let Q be the forcing adding a random real, i.e., $\{p : p$ a measurable set of reals of positive measure$\}$ and $\underset{\sim}{\leq}^*$, $\underset{\sim}{h}$ be a Q-name, $p \in Q$, $p \Vdash_Q$ "$\underset{\sim}{\leq}^*$, $\underset{\sim}{h}$ satisfies $(*)$". Let $\underset{\sim}{R} = \{(\alpha_1, \alpha_2, \alpha_3\} \subseteq [\omega, \omega_1] : \underset{\sim}{h}(\alpha_0, \alpha_1) = \underset{\sim}{h}(\alpha_0, \alpha_2) = \underset{\sim}{h}(\alpha_1, \alpha_2)\}$. Define a forcing notion

$$P = \{(q, W) : q \in Q \text{ has measure } > \tfrac{1}{2},\ W \subseteq \omega_1 \text{ is finite,}$$
$$\text{and for every } t \subseteq W,\ |t| = 3,\ q \Vdash_Q \text{“} t \notin \underset{\sim}{R} \text{”}\}.$$

It is enough to prove P satisfies the countable chain condition. For then, for every $\alpha < \omega_1$, $p_\alpha = ([0,1], \{\alpha\}) \in P$, hence there is $p^* \in P$, $p^* \Vdash_P$ "for unboundedly many $\alpha < \omega_1$, p_α is in the generic subset of P". (Otherwise there is a P-name $\underset{\sim}{\alpha}$ of the bound, and by the countable chain condition it has $\aleph_0$ possible values, α_n $(n < \omega)$, but $p_{\cup_n \alpha_n + 1}$ gives a contradiction.) Let $D_\alpha =$

$\{p \in P : (\exists \beta)(\alpha < \beta < \omega_1 \wedge p_\beta \leqq p\}$, it is dense above p^*, hence by MA there is a directed $G \subseteq P$, $p^* \in G$, $G \cap D_\alpha \neq \emptyset$ for every $\alpha < \omega_1$ and, w.l.o.g., $|G| = \aleph_1$.

Let $q^* = \cap \{q :$ for some $W, (q, W) \in G\}$, as G is directed and $(q, W) \in G \Rightarrow q$ has measure $> \frac{1}{2}$; by MA, q^* is measurable and has measure $\geqq \frac{1}{2}$. Let $W^* = \{\alpha :$ for some $(q, W) \in G$, $p_\alpha \leqq q^*\}$, as $G \cap D_\alpha \neq \emptyset$, $W^* - \alpha \neq \emptyset$, hence $W^* \subseteq \omega_1$ is unbounded. It is also clear that for $t \subseteq W^*$, $q^* \Vdash_Q$ "$t \not\subseteq \underset{\sim}{R}$" (as G is directed, $t \subseteq W$, $(q, W) \in G$, for some q, W; $q \Vdash_Q$ "$t \not\subseteq \underset{\sim}{R}$" by P's definition, but $q^* \leqq q$). So this contradicts (b) of 3.1.

So suppose $(q_i, W_i) \in P$ for $i < \omega_1$, and it suffices to find two compatible ones to finish the proof. We can replace $\{(q_i, W_i) : i < \omega_1\}$ by any uncountable subset.

Let $W_i = \{\alpha^i_0, \dots, \alpha^i_{k-1}, \alpha^i_k, \dots, \alpha^i_{e-1}\}$ (w.l.o.g. k, e does not depend on i), and $\alpha^i_l = \alpha^0_l$ for $l < k$, $\alpha^i_0 < \cdots < \alpha^i_{k-1} < \alpha^i_k < \cdots < \alpha^i_{e-1}$ and $\alpha^i_{e-1} < \alpha^j_k$ for $i < j$. We know that (after forcing with Q) there are $a^i_0, \dots, a^i_{e-1}$, such that $a^i_l <^* \alpha^i_l$, the a^i_l are pairwise $<^*$-incomparable. Hence there are Q-names $\underset{\sim}{a}^i_l$ for them. In Q there are pairwise disjoint $q^n_i \geqq q_i$, $q_i = \bigcup_{n<\omega} q^n_i$ and $q^n_i \Vdash_Q$ "$\langle \underset{\sim}{a}^i_0, \dots, \underset{\sim}{a}^i_{e-1}\rangle = \langle a^{i,n}_0, \dots, a^{i,n}_{e-1}\rangle$". Also there is $n(i)$ and rationals u^n_i such that q^n_i has measure $> u^n_i$, for $n < n(i)$ and $\Sigma_{n<n(i)} u^n_i > \frac{1}{2}$. So w.l.o.g. $n(i)$, $a^{i,n}_0, \dots, a^{i,n}_{e-1}$, u^n_i for $n < n(i)$, does not depend on i and are $n(*)$, $a^n_0, \dots, a^n_{e-1}$ $(n < n(*))$, u^n. Also, because of MA, there are w.l.o.g. q^n_*, $q^n_* \subseteq q^n_i$ for every i, q^n_* has measure $\geqq u^n$ (do this for each n successively; of course, replace our set of conditions by an uncountable subset).

Let $q_* = \bigcup_{n<n(*)} q^n_*$, so clearly $q_* \subseteq q_i$ hence $q_* \geqq q_i$ (in Q) and q_* has measure $\geqq \Sigma\, u^n > \frac{1}{2}$ (as the q^n_* remains pairwise disjoint). Now $(q_*, W_1 \cup W_2) \in P$, the $a^n_0, \dots, a^n_{e-1}$ exemplify this; and it is $\geqq (q_1, W_1) = p_1$, $(q_2, W_2) = p_2$, so we finish.

§4. An attempt on "every set of reals has the Baire property"

The following is a good introduction to the measure case.

4.1. ATTEMPTED MAIN THEOREM. *If every Σ^1_3 set of reals has the property of Baire (i.e., outside a set of the first category, it is equal to an open set), then $\aleph_1$ is an inaccessible cardinal in the constructible universe L.*

4.1A. REMARK. A Σ^1_3 set of reals is a set of the form $\{x : \exists y\, \forall z\, \exists w \varphi(x, y, z, w, a)\}$ where x, y, z, w vary over reals (i.e., members of ${}^\omega 2$), a is a real, and φ an arithmetical formula.

Henceforth we assume that the hypothesis of the theorem holds, but that the conclusion fails, and eventually get a contradiction. So for some real a^*, $\aleph_1^{L[a^*]} = \aleph_1$ (i.e., $\aleph_1$ in the universe $L[a^*]$ is $\aleph_1$ of the true universe V) (this is because $\aleph_1$, being regular in V, is regular in L, but is not inaccessible by an assumption, hence is a successor, $\aleph_1^V = (\mu^+)^L$, so a^* can be any real which codes a well-ordering of ω of order-type μ).

The proof is broken into a series of lemmas and definitions, which lead to the construction of two disjoint Σ_3^1 sets of reals, each nowhere of the first category (i.e., in no open set), thus getting the desired contradiction.

4.1B. Notation. For $A \subseteq {}^{\omega}2$, $\eta \in {}^{\omega>}2$, let $A_{[\eta]} = \{\nu \in A : \eta \triangleleft \nu\}$.

The following forcing notion plays a central part in our proof.

4.2. Definition. UM (universal meagre) is the following forcing notion:

(a) Its set of elements is: $\{(t, T) : T \subseteq {}^{\omega>}2$ is a perfect tree, $t = T \restriction n$ for some $n\}$, where $T \restriction n = \{\eta \in T : l(\eta) \leqq n\}$. If $t = T \restriction n$ for some perfect T, we say $\mathrm{ht}(t) = n$, t a tree of height n.

(b) $(t_0, T_0) \leqq (t_1, T_1)$ if $t_0 = t_1 \restriction \mathrm{ht}(t_0)$, $T_0 \subseteq T_1$.

4.3. Sketch of the Proof. UM is the natural forcing for making the union of all old closed nowhere-dense sets, a set of the first category (see below). In fact, the natural approach to prove the conjecture "if ZFC is consistent then "ZFC + every Σ_3^1 set of reals has the Baire property" is consistent" is as follows. We use iterated forcing $\bar{Q} = \langle P, Q_\alpha : \alpha < \alpha_0 \rangle$ such that, for unboundedly many α's and for every $\alpha < \beta < \alpha_0$ and P_β-names $\underset{\sim}{r}_e$ $(e = 1, 2)$ of reals generic over V^{P_α}, $P_{\alpha+1} = P * \underset{\sim}{\mathrm{UM}}$ ($\underset{\sim}{\mathrm{UM}}$, i.e., UM as interpreted in V^{P_α}), and for some $\gamma \geqq \beta$,

$$P_{\gamma+1} = P_\gamma \underset{P_\alpha + \underset{\sim}{r}_1 = \underset{\sim}{r}_2}{*} P_\gamma,$$

i.e., two copies of P_γ amalgamated over P_α and $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$; more formally it is $\{(p_1, p_2) : p_1 \in P_\gamma, p_2 \in P_\gamma$, and for every $q \in P_\alpha$, and finite function f from ω to 2, for $l = 1, 2$; *if* some $p_l' \geqq p_l$, $p_l' \geqq q$ and $p_l' \Vdash$ "$f \subseteq \underset{\sim}{r}_l$" *then* for some $p_{3-l}' \geqq p_{3-l}$, $p_{3-l}' \geqq q$ and $p_{3-l}' \Vdash f \subseteq \underset{\sim}{r}_{3-l}\}$.

Such an approach was tried (at least for the parallel case of measure) by Truss (the Baire property was considered "the little sister"). The problem was to show that iteration satisfies the countable chain condition (in order to show that it does not collapse $\aleph_1$).

If V satisfies $\mathrm{MA} + 2^{\aleph_0} > \aleph_1$, then UM satisfies: among any $\aleph_1$ conditions, there are $\aleph_1$ which are below one condition. This is a strong strengthening of the countable chain condition. This condition is preserved by the amalgamation (as above), but if we try then to force by UM again, it is no longer over a model which satisfies MA again, hence it is not obvious why it should satisfy the strengthening of the countable chain condition (and this is why Truss had asked "does adding a Cohen generic preserve MA?" which Roitman answered negatively (for more on history see Harrington and Shelah [1]).

We first show that (if the theorem fails, as we assume) for every real a we can force with UM over $L[a]$ (i.e., there is a directed $G \subseteq \mathrm{UM}^{L[a]}$, which meets every dense subset of $\mathrm{UM}^{L[a]}$ which belongs to $L[a]$). Then we shall construct a special function $h^*:[\omega_1]^2 \to \omega$ ($[A]^2$ — the set of increasing pairs from A), so every real a induces a colouring of $[\omega_1]^2$ by red ($=0$) and green ($=1$): the colour of $\langle i,j\rangle$ is $a(h^*(i,j))$ (note $\aleph_1^{L[a^*]}$ is $\aleph_1^V$). Now call a real a red [green] if ω_1 is the union of $\aleph_0$ sets A_n ($n<\omega$) so that each A_n is homogeneously red [green]. Clearly the red set ($=$ the set of red reals) is disjoint from the green set. Now together with h^*, we shall construct some names in the forcing $\mathrm{UM} * \underset{\sim}{\mathrm{UM}}$ (i.e., we force by UM, and in the universe we get we force by UM again). There will be two sets of names: the red and the green. There is a name $\underset{\sim}{a}^{\mathrm{rd}}$ which is forced to be a generic real (i.e., no closed nowhere-dense sets from the ground model), and names, for $i<\omega_1$, $\underset{\sim}{n}^{\mathrm{rd}}(i)$ of natural numbers, so that $\{i : \underset{\sim}{n}^{\mathrm{rd}}(i)=n\}$ is (forced to be) homogeneously red for the colouring $\underset{\sim}{a}^{\mathrm{rd}}(h*(-,-))$, hence $\underset{\sim}{a}^{\mathrm{rd}}$ is a red real. Similarly there are green $\underset{\sim}{a}^{\mathrm{gr}}$, $\underset{\sim}{n}^{\mathrm{gr}}(i)$.

Now for every real a, we can force by $\mathrm{UM} * \underset{\sim}{\mathrm{UM}}^{L[a^*,a]}$ and then interpreting $\underset{\sim}{a}^{\mathrm{rd}}$, $\underset{\sim}{a}^{\mathrm{gr}}$ get red and green reals which belong to no closed nowhere-dense set from $L[a^*,a]$. With a little more care we get them in any open interval. Now, any first category set in V is included in the union of countably many closed nowhere-dense sets $\mathrm{Lim}\, T_n$ (T_n a perfect tree $\subseteq {}^{\omega>}2$). For some a, $\langle T_n : n<\omega\rangle \in L[a^*,a]$. Hence, by the above, the red set and the green set are every- where not of the first category, and they are disjoint, so they do not satisfy the Baire property. Really, we replace them by some Σ_3^1 subsets. Most of our efforts will be to construct h^* and the names simultaneously by finite approximations.

NOTATION. If $V_1 \subseteq V_2$ are universes, $B \in V_1$ a Borel set of reals, then B^{V_2} is a Borel set in V_2 having the same definition.

The following is well known.

4.4 LEMMA. *Suppose a is a real, then* $A = \bigcup \{(\mathrm{Lim}\, T)^V : T \in L[a]$ *a perfect tree*$\}$ *is of the first category.*

PROOF. Suppose not.

Clearly A is a Σ_2^1 set of reals ($x \in A \Leftrightarrow (\exists T)(x \in \mathrm{Lim}\, T$ and T is constructible from a) and $x \in \mathrm{Lim}\, T$ is Σ_0^1 and "T constructible from a" is Σ_2^1). So by our assumption it is equal to an open set outside a first category set. A is not of the first category, hence in some interval $({}^{\omega}2)_{[\eta]}=\{\nu\in{}^{\omega}2:\eta\lhd\nu\}$ its complement is of the first category. As A is invariant under finite changes, ${}^{\omega}2-A$ is of the first category. Now in $L[a]$ there is a Σ^1_2 quasi-linear ordering $<^*$ of A, with every initial segment being of the first category (there is such an ordering of the perfect $T\in L[a]$, $\langle T_i:i<\aleph_1\rangle$, so $x\leqq^* y$ if and only if $\forall\alpha\,[y\in\bigcup_{i<\alpha}T_i\Rightarrow x\in\bigcup_{i<\alpha}T_i]$); more exactly: $(\exists R)[(\omega,R)$ is isomorphic to some $L_\alpha[a]$, $\alpha<\omega_1$ and for some perfect $T\in L_\alpha[a]$, $x\in\lim T$ but $y\notin\operatorname{Lim}T'$ for every $T'\in L_\alpha[a]]$.

Let $B=A\times A$,

$$B_0=\{(x,y):x\in A,\ y\in A,\ x\leqq^* y\},$$

$$B_1=\{(x,y):x\in A,\ y\in A,\ y\leqq^* x\}.$$

Clearly $B=B_0\cup B_1$ is a subset of ${}^{\omega}2\times{}^{\omega}2$, with complement of the first category. For every $y\in{}^{\omega}2$, $\{x:(x,y)\in B_0\}$ is of the first category, so if B_0 has the Baire property, then by the analog to Fubini theory, B_0 is of the first category. Similarly $\{y:(x,y)\in B_1\}$ is of the first category for each $x\in{}^{\omega}2$, so if B_1 has the Baire property, then it is of the first category. But $B_0\cup B_1=B$ is not of the first category. So for some $e=0,1$, B_e does not have the Baire property; however, it is a Σ^1_2 set, contradicting an assumption (formally, we should translate the situation from ${}^{\omega}2\times{}^{\omega}2$ to ${}^{\omega}2$, which is trivial).

4.5. Lemma. *For every real a, there is a generic set for* $\mathrm{UM}^{L[a]}$ *(over* $L[a]$*).*

Proof. By 4.4 there is $B\supseteq\bigcup\{B^V:B\in L[a]$ a Borel set of the first category$\}$, B a Borel set of the first category. In $L[a,B]$, $B\subseteq\bigcup_{n<\omega}B_n$, B_n nowhere dense, and so there is a countable family H of nowhere-dense subsets of ${}^{\omega}2$ (in $L[a,B]$) so that every nowhere-dense subset of ${}^{\omega}2$ in $L[a]$ is included in a member of the family (you have to work a little: Note that if $B\in L[a]$ is nowhere dense and closed, then there is a perfect $T\subseteq{}^{\omega>}2$, $\operatorname{Lim}T=B$ and (in $L[a]$) there is a perfect T', $T\subseteq T'$ such that for every $\eta\in T'$ for some $\nu=\nu_\eta\in{}^{\omega>}2$, $\eta\lhd\nu\in T'$ and $(\forall\rho)[\rho\in T\Rightarrow\nu{}^\wedge\rho\in T']$. Now $\operatorname{Lim}T'\subseteq\bigcup_{n<\omega}B_n$, hence for some $\eta\in T'$, and $n<\omega$, $(\operatorname{Lim}T')_{[\eta]}\subseteq B_n$. Hence $\operatorname{Lim}T\subseteq\{\rho_\rho:\nu_\eta{}^\wedge\rho\in B_n\}$, so the family $\{\{\rho:\nu{}^\wedge\rho\in B_n\}:n<\omega,\ \nu\in{}^{\omega>}2\}$ suffices). Let N be a countable transitive elementary submodel of $L_{\aleph_1}[a,B]$ to which H belongs.

We can find a generic object for $\mathrm{UM}^{L[a,B]}\cap N$ over $L[a,B]$ as this is a countable forcing, hence equivalent to forcing a generic real. The generic set induces a generic set of $\mathrm{UM}^{L(a)}$.

Unfortunately, we ran into difficulties trying to build h^* and the names.

§5. On "every set of reals is measurable"

5.1. MAIN THEOREM. *If every* Σ^1_3 *set of reals is measurable, then* $\aleph_1$ *is an inaccessible cardinal in* L.

REMARKS.

(1) The theorem is proved in ZFC, of course. However, very little use of the axiom of choice is made, only that $\aleph_1$ is not singular in any $L[a]$, a a real. For this it suffices that $\aleph_1$ is regular, which follows from the countable axiom of choice (i.e., the existence of choice for a family of countably many sets).

(2) In the proof we use, in fact, only two formulas: $\varphi(x,y)$, $\psi(x,y)$. For the first we need, as a parameter, any real a such that $\bigcup\{B : B$ a Borel set of measure zero which has a code in $L[a]\}$ does not have measure zero (i.e., for such a, $\{x : \varphi(x,a)\}$ is a non-measurable set of reals). For the second we need to assume there is no a as above, and use as a parameter any a such that $\aleph_1^{L[a]} = \aleph_1$.

(3) It is known that there is a generic extension of L not collapsing cardinals nor violating CH, in which every definable (with no parameter!) set of reals is measurable, e.g., force by $\Phi = \{P : P_{\varphi,\psi}$ (see Definition 6.2) and it satisfies the ccc where φ is Σ^1_2, ψ is $\Sigma^1_1\}$ with the order $<$ (being a complete Boolean subalgebra). Note that if P_1, $P_2 \in \Phi$ then $P_1 \times P_2 \in \Phi$ and P_1, $P_2 < P_1 \times P_2$ (force MA + not CH and use absoluteness). After forcing with Φ we get, as a generic object (its union, more exactly), a ccc forcing notion, with which we force.

5.1A. CONCLUSION. The following are equiconsistent:

(1) ZFC + there is an inaccessible cardinal.

(2) ZFC + every Σ^1_3 set of reals is measurable.

(3) ZFC + every set of reals defined by a first-order formula with real and ordinal parameters is measurable.

(4) ZF + the axiom of countable choice + every set of reals is measurable.

(5) ZF + DC + every set of reals is measurable.

(Solovay proved (1) $\Rightarrow$ (2), (3), (4), (5), our main theorem is (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (2) is trivial, and (4) $\Rightarrow$ (1) is Remark (1) above. Lastly (5) $\Rightarrow$ (4) is trivial.

By minor changes in the proof we can get

5.1B. THEOREM. (ZF + DC) *If there is a set of* $\aleph_1$ *reals, then there is a non-measurable set of reals.*

REMARK. The parallel theorem on the Baire property is not provable in ZF + DC, by §7.

PROOF OF THEOREM 5.1. Henceforth we assume the hypothesis of the theorem, and that the conclusion fails, so for some real a^*, $\aleph_1^{L[a^*]} = \aleph_1$ (see 4.1A). First we note the following, which is well known.

5.2. LEMMA. *Suppose a is a real, B_i ($i < \aleph_1^{L[a]}$) a list of all Borel sets of measure zero of $L[a]$, and let B_i^V be the Borel sets with the same definition in V. Then obviously each B_i^V has measure zero, moreover, $\bigcup_i B_i^V$ has measure zero.*

PROOF. $\bigcup_i B_i^V$ is measurable because it is defined by a $\Sigma_2^1[a]$ formula; moreover, on it there is a Σ_2^1 quasi-linear ordering $<^*$, with every initial segment of measure zero. This contradicts the Fubini theorem for $\{(x, y) : x, y \in \bigcup_i B_i, x <^* y\}$, except when $\bigcup_i B_i^V$ has measure zero (see the proof of 4.4 for more details).

5.3. SCHEME OF THE PROOF. We shall construct a special function $h^* : [\aleph_1]^2 \to \omega$ ($[A]^2$ — the set of increasing pairs from A). So for every real a (which is a function from ω to 2) a colouring of $[\aleph_1]^2$ by red ($= 0$) and green ($= 1$) is defined: $\langle i, j \rangle$ is coloured by a $(h^*(i, j))$ (note $\aleph_1$ is the true $\aleph_1$, that of V). Now we consider the following two sets of reals: The red [green] set is the set of reals a, such that: $\aleph_1$ is the union of $\aleph_0$ sets, A_n ($n < \omega$), so that each A_n is homogeneously red (i.e., for $i < j \in A_n$, the colour is red) [each A_n is homogeneously green] for the coloring which corresponds to a. Those two sets are disjoint, and if h^* is "simply" defined, they hopefully will be "simply" defined too. We could also use other properties, like: an $\aleph_1$-tree defined from the real a has an ω_1-branch, or is special. So we have to construct a suitable h^*, and prove that both sets of reals have outer measure 1 where, for technical reasons, we work with 2^ω, as the set of reals with the measure LbMs, where LbMs $((2^\omega)_{[\eta]}) = 2^{-l(\eta)}$, $(2^\omega)_{[\eta]} = \{\nu \in {}^\omega 2 : \eta \triangleleft \nu\}$. Note that every closed set $A \subseteq 2^\omega$ is uniquely determined by $T[A] = \{\eta \upharpoonright l : l < \omega, \eta \in A\}$, which is a closed subtree of $2^{<\omega}$, as $A = \operatorname{Lim} T$.

The natural forcing, which makes the union of all old Borel sets of measure zero into a measure-zero set, is the amoeba forcing Am: the set of conditions is the family of measurable sets of measure $< \frac{1}{2}$; w.l.o.g. we consider only open sets. For technical reasons we use their complements, i.e., $\mathrm{Am}' = \{T : T \subseteq 2^{<\omega}$ a closed tree, LbMs $(\operatorname{Lim} T) > \frac{1}{2}\}$. So the generic object of Am′ is, essentially, a closed tree $T \subseteq 2^{<\omega}$, LbMs $(\operatorname{Lim} T) = \frac{1}{2}$, $\operatorname{Lim} T$ disjoint to all B^V, for B an "old" measure-zero Borel set. Now in the forcing $\mathrm{Am}' * \underset{\sim}{\mathrm{Am}}'$ (we iterate the forcing), we shall build (in $L[a^*]$) two systems of names: the red system and the green system. Both give a name $\underset{\sim}{a}^{\rm rd}[\underset{\sim}{a}^{\rm gn}]$ of a real, which is inside the generic tree of the first Am$'$, and $h^*:[\aleph_1]^2\to\omega$ and for each $i<\aleph_1$ a natural number $\underset{\sim}{n}^{\rm rd}(i)$ $[\underset{\sim}{n}^{\rm gn}(i)]$,[†] such that $\underset{\sim}{A}^{\rm rd}_m=\{i:\underset{\sim}{n}^{\rm rd}(i)=m\}$ is homogeneously red (similarly for green), i.e., if $\underset{\sim}{n}^{\rm rd}(i)=\underset{\sim}{n}^{\rm rd}(j)$ then $\underset{\sim}{a}^{\rm rd}(h^*(i,j))=0$. So for any real a, we force by $(\mathrm{Am}'*\underset{\sim}{\mathrm{Am}}')^{L[a^*,a]}$. So, e.g., for red, we get a real a', such that the colouring it induces on $[\aleph_1]^2$ is the union of countably many homogeneously red sets, so it belongs to the red set. Also, it belongs to the generic tree of the first forcing, hence it does not belong to any B^V, $B\in L[a^*,a]$ a Borel set of measure zero. Hence a' is a random real over $L[a^*,a]$. Being careful a little more in the details we then prove that the "red set" has outer measure 1. As the same holds for the "green" set, we finish. In fact, we do not use the red and green sets, but definable subsets of them.

Unfortunately, we have not proved that for every a, there is a tree generic for Am$'$ over $L[a^*,a]$. We have been able to prove (by Lemma 5.2, see below) that there is a closed tree T such that Lim T has measure $\frac{1}{2}$ and is disjoint to B^V for any Borel set of measure zero $B\in L[a^*,a]$. This may look like "the poor man generic tree for Am$'$" and we call it by this name, but it will suffice. Note that

5.3A. Fact. If T is such a tree, $A_l\subseteq 2^\omega$ open $(l<\omega)$, $A_{l+1}\subseteq A_l$, and $\bigcap_l A_l$ has measure zero, $\langle A_l:l<\omega\rangle$ defined in $L[a^*,a]$, *then* for some $\eta\in T$ and $l<\omega$, $(\mathrm{Lim}\,T)_{[\eta]}\cap A_l=\emptyset$. (Really each A_l defined in $L[a^*,a]$ suffices.)

So choosing the first pair $\langle\eta,l\rangle$ for which this occurs, we obtain essentially a "name" of a member of $2^{<\omega}\times\omega$; we use such names for the $\underset{\sim}{n}^{\rm rd}(i)$, $\underset{\sim}{n}^{\rm gn}(i)$.

We now proceed to the actual proof.

5.4. Definition. We define the natural number $\mu(k)$, for $k<\omega$, as follows (they should be just increasing fast enough): $\mu(k)=2^{2^{2^{5k+7}}}$.

5.5. Definition. For a closed tree T, the function ms_T is defined, for $\eta\in{}^{\omega>}2$, by $ms_T(\eta)=\mathrm{LbMs}\,[(\mathrm{Lim}\,T)\cap(2^\omega)_{[\eta]}]$. For a function $m:T\to\mathbf{Q}$, T is suitable for m if $m(\eta)\leqq ms_T(\eta)$ for every $\eta\in{}^{\omega>}2$.

5.6. Claim. Let B be a set of measure zero, then there is a perfect tree T and function $m:T\to\mathbf{Q}$ ($\mathbf{Q}$ — the rational numbers) such that:

(a) $(\mathrm{Lim}\,T)\cap B=\emptyset$, $m=ms_T$,

(b) $ms_T(\langle\ \rangle)=\frac{1}{2}$, and $ms_T(\eta)$ has the form $k/4^{l(\eta)+1}$, $0\leqq k<4^{l(\eta)+1}$, and $k\neq 0$ iff $\eta\in T$.

[†] Both are $\mathrm{Am}*\underset{\sim}{\mathrm{Am}}$-names.

A similar lemma is:

5.7. CLAIM. Let B be a set of measure zero. Then there is a perfect tree T and $m : T \to \mathbf{Q}$ and natural numbers $n(k)$ $(k < \omega)$ such that:

(a) $(\operatorname{Lim} T) \cap B = \emptyset$, $m = ms_T$,

(b) $ms_T(\langle\ \rangle) = \frac{1}{2}$ and for every $\eta \in T$, $l(\eta) \leqq n(k)$, $ms_T(\eta)$ is $l/4^{n(k)+1}$, $0 < l < 4^{n(k)+1}$,

(c) for every $\eta \in {}^{n(k)}2 \cap T$, $ms_T(\eta) > 2^{\ n(k)}\ (1 - 1/\mu(k))$.

PROOF OF 5.6. As B has measure zero, there is an open set A, $B \subseteq A$, LbMs $(A) < 1/100$. Now we define by induction on $n < \omega$ a set $t_n \subseteq {}^n2$, and a function $m_n : t_n \to \mathbf{Q}$ and an open set A_n such that:

(*) (1) if $\eta \in t_n$, then $4^{n+1} m_n(\eta)$ is a natural number $< 4^{n+1}$ (but > 0) and $m_n(\eta) + 1/4^{n+1} < \operatorname{LbMs}((2^\omega - A_n)_{[\eta]}) < m_n(\eta) + 1/4^{n+1} + 1/4^{n+1}$, hence necessarily $m_n(\eta) < 2^{-l(\eta)}$;

(2) $A_{n-1} \subseteq A_n$ are open, $A_0 = A$;

(3) if $\eta \in t_n$ then $m_{n+1}(\eta\hat{\ }\langle 0\rangle) + m_{n+1}(\eta\hat{\ }\langle 1\rangle) = m_n(\eta)$ (of course, if $\nu \in {}^n2 - t_n$, $m_n(\nu) = 0$).

For $n = 0$, $t_0 = \{\langle\ \rangle\}$, $m_0(\langle\ \rangle) = \frac{1}{2}$.

For $n + 1$, for each $\eta \in t_n$, $l \in \{0, 1\}$, choose a maximal integer $k(\eta, l)$ such that

$$k(\eta, l)/4^{n+2} < \operatorname{LbMs}((2^\omega - A_n)_{[\eta\hat{\ }\langle l\rangle]}).$$

So

$$\operatorname{LbMs}((2^\omega - A_n)_{[\eta\hat{\ }\langle l\rangle]}) - k(\eta, l)/4^{n+2} \leqq 1/4^{n+2}.$$

Hence

$$\begin{aligned}
&\operatorname{LbMs}((2^\omega - A_n)_{[\eta]}) - (k(\eta,0) + k(\eta,1))/4^{n+2} \\
&\quad = (\operatorname{LbMs}((2^\omega - A_n)_{[\eta\hat{\ }\langle 0\rangle]}) - k(\eta,0)/4^{n+2}) + (\operatorname{LbMs}((2^\omega - A_n)_{[\eta\hat{\ }\langle 1\rangle]}) - k(\eta,1)/4^{n+2}) \\
&\quad \leqq 1/4^{n+2} + 1/4^{n+2} = 2/4^{n+2}.
\end{aligned}$$

Hence

$$\operatorname{LbMs}((2^\omega - A_n)_{[\eta]}) \leqq (k(\eta,0) + k(\eta,1) + 2)/4^{n+2}.$$

But by (*)(1)

$$m_n(\eta) + 4/4^{n+2} < \operatorname{LbMs}((2^\omega - A_n)_{[\eta]}).$$

Together we get

$$m_n(\eta) < (k(\eta,0) + k(\eta,1) - 2)/4^{n+2}.$$

First assume $k(\eta,0)$, $k(\eta,1) > 0$.

As $4^{n+1}m_n(\eta)$ is a natural number we can choose natural numbers $k'(\eta,0)<k(\eta,0)$, $k'(\eta,1)<k(\eta,1)$ such that

$$m_n(\eta)=(k'(\eta,0)+k'(\eta,1))/4^{n+2}.$$

We then let $m_{n+1}(\eta\,^\wedge\langle l\rangle)=k'(\eta,l)/4^{n+2}$. If $k(\eta,l)\leqq 0$ we can check that $k(\eta,1-l)>0$, and we choose $k'(\eta,1-l)$ such that $m_n(\eta)=k'(\eta,1-l)/4^{n+1}$, and we can check that $k'(\eta,1-l)\leqq k(\eta,1-l)$, and we let

$$m_{n+1}(\eta\,^\wedge\langle 1-l\rangle)=k'(\eta,1-l)/4^{n+1}=m_n(\eta).$$

We let $t_{n+1}=\{\nu\in 2^{n+1}: m_{n+1}(\nu)$ is defined and positive$\}$. Now one of the inequalities in $(*)_{n+1}(1)$ holds, with A_n instead of A_{n+1}.

We prove the left inequality of $(*)(1)$.

Suppose $\eta^\wedge\langle l\rangle\in t_{n+1}$. First, if $k(\eta,0)$, $k(\eta,1)>0$ then

$$\begin{aligned} m_{n+1}(\eta^\wedge\langle l\rangle)+1/4^{n+2} &= k'(\eta^\wedge\langle l\rangle)/4^{n+2}+1/4^{n+2}\\ &\leqq k(\eta,l)/4^{n+1}\\ &<\mathrm{LbMs}((2^\omega-A_n)_{[\eta^\wedge\langle l\rangle]}). \end{aligned}$$

Second, if the condition "$k(\eta,0)$, $k(\eta,1)>0$" fails, then necessarily $k(\eta,1-l)\leqq 0$, and by our choice above

$$\begin{aligned} m_{n+1}(\eta^\wedge\langle l\rangle)+1/4^{n+2} &= m_n(\eta)+1/4^{n+2}\\ &=(m_n(\eta)+1/4^{n+1})-3/4^{n+2}\\ &<\mathrm{LbMs}((2^\omega-A_n)_{[\eta]})-3/4^{n+2}\\ &=\mathrm{LbMs}((2^\omega-A_n)_{[\eta^\wedge\langle l\rangle]})\\ &\quad+(\mathrm{LbMs}((2^\omega-A_n)_{[\eta^\wedge\langle 1-l\rangle]}-3/4^{n+2}\\ &\leqq \mathrm{LbMs}((2^\omega-A_n)_{[\eta^\wedge\langle l\rangle]}). \end{aligned}$$

The last inequality holds as otherwise

$$\mathrm{LbMs}((2^\omega-A_n)_{[\eta^\wedge\langle 1-l\rangle]})>3/4^{n+2},$$

but as $k(\eta,1-l)=0$, by the choice of $k(\eta,1-l)$,

$$\mathrm{LbMs}((2^\omega-A_n)_{[\eta^\wedge\langle 1-l\rangle]})\leqq 1/4^{n+2}.$$

Contradiction.

We can increase A_n to A_{n+1} so that both hold.

Now $T=\bigcup_n t_n$ is the required tree, and $m_n(\eta)=ms_T(\eta)$ for $\eta\in t_n: T$ is $\subseteq {}^{\omega>}2$ and is a closed tree. As $B\subseteq A_0\subseteq A_n$, A_n open, Lim T closed and for no

$\eta \in T$, $({}^{\omega}2)_{[\eta]} \subseteq A_0$ (by $(*)(1)$), clearly $(\operatorname{Lim} T) \cap B = \emptyset$, in fact $(\operatorname{Lim} T) \cap A_n = \emptyset$.

For any $\eta \in T$ and k, $l(\eta) \leqq k < \omega$

$$\begin{aligned} m_n(\eta) &= \sum \{m_n(\nu) : \nu \in (t_k)_{[\eta]}\} \\ &< \sum \{\mathrm{LbMs}((2^{\omega} - A_k)_{[\nu]}) + 1/4^{k+1} + 1/4^{k+1} : \nu \in (t_k)_{[\eta]}\} \\ &\leqq \sum \{\mathrm{LbMs}((2^{\omega})_{[\nu]}) + 2/4^{k+1} : \nu \in (t_k)_{[\eta]}\} \\ &= \sum \{1/2^k + 2/4^{k+1} : \nu \in (t_k)_{[\eta]}\} \\ &= |t_k|/2^k + 2|t_k|/4^{k+1} \\ &\leqq |t_k|/2^k + 2 \cdot 2^k/4^{k+1} \\ &= |t_k|/2^k + 1/2^{k+1}. \end{aligned}$$

When $k < \omega$ increase, the first term converges to LbMs $(\operatorname{Lim} T)$ and the second term converges to zero. So we can conclude that $m_n(\eta) \leqq ms_T(\eta)$ for every $\eta \in T$.

On the other hand, for every n, by $(*)(1)$,

$$\mathrm{LbMs}(({}^{\omega}2)_{[\eta]} - A_n) - 2/4^{n+1} < m_n(\eta) \leqq ms_T(\eta) \leqq \mathrm{LbMs}(({}^{\omega}2)_{[\eta]} - A_n)$$

(as $(\operatorname{Lim} T) \cap A_n = \emptyset$), hence $0 \leqq ms_T(\eta) - m_n(\eta) < 2/4^{n+1}$. Moreover, for every $k < \omega$

$$\begin{aligned} ms_T(\eta) - m_n(\eta) &= \sum \{ms_T(\nu) - m_k(\nu) : \nu \in t_k, \eta \lhd \nu\} \\ &\leqq |t_k| \cdot 2/4^{k+1} \leqq 2^k \cdot 2/4^{k+1} = 1/2^{k+1}. \end{aligned}$$

As this holds for every $k > n$, $m_T(\eta) = m_n(\eta)$.

Proof of 5.7. The proof is similar, but we define, by induction on k, $n(k)$, $\{t_l : l \leqq n(k), l > n(k')$ for every $k' < k\}$, and the function m_l $(l \leqq n(k))$, and A_k.

5.8 Definition. (1) Let N_n be the set of pairs (t, m) such that:

(a) t is a non-empty subset of ${}^{n\geqq}2$, closed under initial segments, and for every $\eta \in t \cap {}^{n>}2$, for some l, $\eta\,\hat{}\,\langle l \rangle \in t$.

(b) m is a function from t to the rationals, $m(\langle\ \rangle) = \frac{1}{2}$, $4^{l(\eta)+1} m(\eta)$ is a natural number >0 and $< 4^{l(\eta)+1} 2^{-l(\eta)}$, and for $\eta \in t \cap 2^{<n}$, $m(\eta) = \Sigma\{m(\eta\,\hat{}\,\langle l \rangle) : \eta\,\hat{}\,\langle l \rangle \in t\}$.

(2) We let $N = \bigcup_n N_n$, we call n the height of (t, m) for $(t, m) \in N_n$, and let $n = \mathrm{ht}(t, m)$. We let $(t', m') = (t, m) \upharpoonright n$ if $t' = t \cap {}^{n \geqq}2$, $m' = m \upharpoonright t'$. On N a tree structure is defined:

$$(t_0, m_0) \leqq (t_1, m_1) \qquad \text{if } (t_0, m_0) = (t_1, m_1) \upharpoonright \mathrm{ht}(t_0, m_0).$$

Note that N_n is the nth level and it is finite.

(3) A closed tree $T \subseteq {}^{\omega >}2$ satisfies (t, m) if $T \cap ({}^{\mathrm{ht}(t,m) \geqq}2) = t$, $ms_T \upharpoonright t = m$.

5.9. Definition. (1) M_k is the set of pairs (t, m) such that the following holds, for some $n = \mathrm{ht}(t, m)$:

(a) t is a non-empty subset of ${}^{n \geqq}2$ closed under initial segments and for $\eta \in t \cap {}^{n>}2$ there is $l \in \{0, 1\}$, $\eta \hat{\ } \langle l \rangle \in t$;

(b) m is a function from t to the rationals which are > 0, < 1, $m(\langle\ \rangle) = \frac{1}{2}$, and $m(\eta) = \Sigma\{m(\eta \hat{\ } \langle l \rangle) : (\eta \hat{\ } \langle l \rangle) \in t\}$ for $\eta \in {}^{n>}2$.

(c) We define $r_l = \mathrm{lev}_l(t, m)$ by induction on l; $r_0 = 0$, and r_{l+1} is the first $r > r_l$ such that:

(*) $r \leqq n$, and for every $\eta \in {}^{r \geqq}2 \cap t$, $4^{r+1} m(\eta)$ is an integer, and for every $\eta \in {}^{r}2 \cap t$, $m(\eta) > 2^{-r}(1 - 1/\mu(l + 1))$.

Now we demand that r_k is defined and is equal to n.

(2) $M_{k,n} = \{(t, m) \in M_k : \mathrm{ht}(t, m) = n\}$,

$$M_{k,<n} = \bigcup_{l<n} M_{k,l}, \quad M_{*,<n} = \bigcup_{k} M_{k,<n}, \quad M = \bigcup_{k} M_k$$

for $(t, m) \in M_k$, $\mathrm{rk}(t, m) = k$.

We define $(t, m) \upharpoonright n$ as in Definition 5.8 but maybe $(t, m) \upharpoonright n \not\in M$; also, the order is defined similarly. So M_k is the kth level of M as a tree, M_k is infinite, but $M_{k,<n}$, $M_{*,n}$ are finite.

(3) We define "T satisfies (t, m)" as in Definition 5.8(3).

Remark. In the following definition the 'green part' is not really needed; but if we use other properties (like: a tree defined from the real is special, as a "red" name and a tree defined from the real has an ω_1-branch as a green name), such a thing will be needed.

5.10. Main Definition. We shall define here what is a finite system [a full system]. The finite systems are approximations to the full system, which consist of the "names" discussed before.

A finite [full] system S consists of the following (when several systems are discussed, S will have an additional superscript or be $S(i)$):

(A) *The common part*: A finite subset W of $\aleph_1$ [the set $W = \aleph_1$], and a number $n(1) < \omega$ [$n(1) = \omega$], and a function h from $[W]^2$ to $n(1) = \{l : l < n(1)\}$, such that for $i_1 < i_2 < i_3$ in W, $h(i_1, i_2) \neq h(i_2, i_3)$.

(B) *The red part*: This consists of (everything should have a first subscript rd, which we omit):

(a) For every $(t, m) \in M_{*, \leqq n(1)}$, a natural number $\lambda(t, m)$, and for every $(t_1, m_1) \in N_{\lambda(t,m)}$, there is a member $\rho(t_1, m_1, t, m)$ of $t \cap {}^{\mathrm{ht}(t,m)}2$.

[*Explanation*: This gives partial information on how red reals are defined from poor man generic trees, (t, m) is an approximation of the first tree, (t_1, m_1) of the second tree, $\rho(t_1, m_1, t, m)$ is an initial segment of the real, passing through the first tree, hence random over $L[a^*, a]$.]

(b) Let $\{\eta_l : l < \omega\}$ be a fixed enumeration of ${}^{\omega >}2$, such that $l(\eta_l) \leqq l$ (its significance will appear in (f) and (c)). For any $(t, m) \in M_{k, \leqq n(1)}$, $l < k$, $j < k$, and $\xi \in W$, there is a finite set $A_{l,j}^{(t,m),\xi}$ of sequences from ${}^{\omega >}2$, each of length $\leqq \lambda(t, m)$, and such that

$$\sum \{1/2^{l(\nu)} : \nu \in A_{l,j}^{(t,m),\xi}\} < 1/2^{l+j}.$$

[*Explanation*: This is part of the name $\underset{\sim}{n}^{\mathrm{rd}}(\xi)$; we let

$$C_{l,j}^{(t,m),\xi} = \{\eta \in {}^{\omega}2 : (\exists \nu \in A_{l,j}^{(t,m),\xi}) \eta \lhd \nu\},$$

so it has measure $< 1/2^{l+j}$, hence $\bigcup_{l<\omega, j>j_0} C_{l,j}^{(t,m),\xi}$ is an open set of measure $< 1/2^{j_0-1}$, hence $\bigcap_{j_0<\omega} \bigcup_{l<\omega, j>j_0} C_{l,j}^{(t,m),\xi}$ has measure zero, so for every poor man generic T (this will be the second one), for some $j < \omega$, $\eta \in T$, $(T)_{[\eta]} \cap (\bigcup_{l<\omega} C_{l,j}^{(t,m),\xi}) = \varnothing$. The name $\underset{\sim}{n}^{\mathrm{rd}}$ could be defined after (c). An approximation to it is: the first pair (η, j) such that $(T_1)_{[\eta]}$ (T_1 — the second poor man generic tree) is disjoint to $\bigcup \{C_{l,j}^{(t,m),\xi} : l < \omega,\ (t, m) \text{ is } T_0 \restriction \mathrm{ht}(t, m)\}$ (T_0 — the first poor man generic tree).]

(c) For every $(t, m) \in M_{k, \leqq n(1)}$ and $\xi \in W$ and $(t(0), m(0)) \in N_{\lambda(t,m)}$ there is a function $f_{(t(0),m(0))}^{(t,m),\xi}$ from $\{\eta_l : l < k\} \times k$ into ω.

[*Explanation*: This function is part of the name we describe in (b), i.e., we shall try to make

$$\begin{aligned} A_{\eta,j,l_1} = \{\xi :\ & f_{(t(0),m(0))}^{(t,m),\xi}(\eta, j) = l_1,\ (t, m) \text{ is in the first poor man generic tree } T_0,\\ & (t(0), m(0)) \text{ is in the second poor man generic tree } T_1,\\ & \eta \in {}^{\omega >}2,\ j < \omega, \text{ and } \mathrm{Lim}\, T_1 \text{ is disjoint to } \textstyle\bigcup_{l<\omega} C_{l,j}^{(t,m),\xi}\} \end{aligned}$$

homogeneously red.

Note that instead of using $\omega_1 = \bigcup_{n<\omega} A_n$ we use

$$\omega_1 = \bigcup \{A_{\eta,j,l_1} : \eta \in {}^{\omega >}2,\ j < \omega,\ l_1 < \omega\}$$

so the name $\underset{\sim}{n}^{\mathrm{rd}}(\xi)$ technically does not appear, but this is just a notational change.]

Now the parts described above should satisfy some conditions.

(d) *Monotonicity for* (a): If $(t_0, m_0) < (t_1, m_1)$ (both in $M_{*, \leqq n(1)}$) then $\lambda(t_0, m_0) < \lambda(t_1, m_1)$; moreover if $(t^l, m^l) \in N_{\lambda(t_l, m_l)}$, $(t^0, m^0) < (t^1, m^1)$, then

$$\rho(t^0, m^0, t_0, m_0) \lhd \rho(t^1, m^1, t_1, m_1).$$

[This will guarantee a^{rd} is well defined.]

(e) *Monotonicity for* (b): If $(t^0, m^0) < (t^1, m^1)$ both in $M_{*,\leq n(1)}$, $A_{l,j}^{(t^0,m^0),\xi}$ is defined, then $A_{l,j}^{(t^0,m^0),\xi} = A_{l,j}^{(t^1,m^1),\xi}$. Also

$$f_{(t_0,m_0)}^{(t^0,m^0),\xi} \subseteq f_{(t_1,m_1)}^{(t^1,m^1),\xi} \qquad \text{if } (t_l, m_l) \in N_{\lambda(t^l,m^l)}, \quad (t_0, m_0) < (t_1, m_1).$$

(f) *The homogeneity consistency condition*: If $(t, m) \in M_{k,\leq n(1)}$ and $\xi < \zeta \in W$, and $h(\xi, \zeta) < \mathrm{ht}(t, m)$ and $(t_1, m_1) \in N_{\lambda(t,m)}$ and $\rho = \rho(t_1, m_1, t, m)$, *then*

(i) $\rho(h(\xi, \zeta)) = 0$ (= the red colour), *or*

(ii) for every $l, j < k$, $j \neq 0$ such that $f_{(t_1,m_1)}^{(t,m),\zeta}(\eta_l, j) = f_{(t_1,m_1)}^{(t,m),\xi}(\eta_l, j)$ there is no perfect tree $T \subseteq {}^{\omega>}2$ which satisfies (t_1, m_1), and $(T)_{[\eta_l]}$ is disjoint to $\bigcup_{\alpha<k} C_{\alpha,j}^{(t,m),\xi}$ and also to $\bigcup_{\alpha<k} C_{\alpha,j}^{(t,m),\zeta}$.

This last phrase is equivalent to: $(t_1)_{[\eta_l]}$ is disjoint to $\bigcup_{\alpha<k} A_{\alpha,j}^{(t,m),\xi}$ and also to $\bigcup_{\alpha<k} A_{\alpha,j}^{(t,m),\zeta}$.

(C) *The green part*: It is defined similarly, only in (f)(i) we replace 0 (= red) by 1 (= green).

5.11. DEFINITION. The order between finite systems is defined naturally (for specific (t,m), $\lambda(t,m)$ remain constant as well as $A_{l,j}^{(t,m),\xi}$, $f_{(t(0),m(0))}^{(t,m),\xi}$, but W and $n(1)$ may increase).

5.12. LEMMA. *The family of finite systems satisfies the countable chain condition.*

PROOF. Suppose $S(\gamma)$ $(\gamma < \omega_1)$ are $\aleph_1$ conditions. Then, in the standard way, we can assume that $S(0)$, $S(1)$ are such that: $n \overset{\mathrm{df}}{=} n(1)^{S(0)} = n(1)^{S(1)}$, $\lambda^{S(0)} = \lambda^{S(1)}$, $\rho^{S(0)} = \rho^{S(1)}$ (both as functions), and there is a function g (one-to-one) from $W^{S(0)}$ onto $W^{S(1)}$ which "maps" $S(0)$ onto $S(1)$ in the natural way and is the identity on $W^{S(0)} \cap W^{S(1)}$.

We want to define a common upper bound S.

We let $W^S \overset{\mathrm{df}}{=} W^{S(0)} \cup W^{S(1)}$, $n(1)^S \overset{\mathrm{df}}{=} n+1$. The function h^S is defined as follows: it extends $h^{S(0)}$ and $h^{S(1)}$, and if $\xi < \zeta \in W^S$, $\xi \in W^{S(l)} \Leftrightarrow \zeta \notin W^{S(l)}$ $(l = 0, 1)$, then $h^S(\xi, \zeta) = n$. For each $(t, m) \in M_{*,\leq n}$ we let $\lambda(t, m)$, $\rho(-,-,t,m)$, be as in $S(0)$ and $S(1)$, and for $\xi \in W^{S(l)}$, $A_{l,j}^{(t,m),\xi}$ and $f_{(t_1,m_1)}^{(t,m),\xi}$ are defined as in $S(l)$ (and there is no contradiction in the definition).

The problem is to define all this for $(t, m) \in M_{*,(n+1)}$, $(t, m) \notin M_{*,\leq n}$. So let $(t, m) \in M_{k+1,\leq(n+1)}$, $(t, m) \notin M_{*,\leq n}$, hence $\mathrm{ht}(t, m) = n + 1$. (Clearly (t, m) is not of height zero, so the k is $\geqq 0$.)

Clearly there is a unique $(t(0), m(0)) < (t, m)$, $(t(0), m(0)) \in M_{k,\leq n}$, $M_{*,\leq n}$. W.l.o.g. we concentrate on the red part.

We first define $\lambda(t, m) = \lambda(t(0), m(0)) + |W^S| + (2k + 1)$; and now comes an important point:

for any $j \leqq k$, we define $\langle A^{(t,m),\xi}_{k,j} : \xi \in W^S \rangle$ as an independent family of subsets of $\{\nu : l(\nu) = \lambda(t,m)\}$ (independent in the probabilistic sense); moreover, this holds "above" each $v \in {}^{\lambda(t(0),m(0))}2$ and

$$|A^{(t,m),\xi}_{k,j}| \,/\, 2^{\lambda(t,m)} = 1/2^{k+j+1}.$$

We can define, for $j \leqq k$, $\langle A^{(t,m),\xi}_{j,k} : \xi \in W \rangle$ in any reasonable way, e.g., as in the previous case. (For $j, l < k$, $A^{(t,m),\xi}_{j,l}$ is determined by (e) of 5.10(B).)

Now we have to define $f^{(t,m),\xi}_{(t_1,m_1)}(\eta_l, j)$ for $(t_1, m_1) \in N_{\lambda(t,m)}$, $j, l < k+1$. For $j, l < k$ this will be $f^{(t(0),m(0)),\xi}_{(t_1,m_1)\restriction\lambda(t(0),m(0))}(\eta_l, j)$ for $S(0)$ or $S(1)$ depending on whether $\xi \in W^{S(0)}$ or $\xi \in W^{S(1)}$, and if both we shall not get a contradiction. For $l = k$ or $j = k$ we have full freedom to define $f^{(t,m),\xi}_{(t(0),m(0))}(\eta_l, j)$ and we define it as a one-to-one function of ξ (possible as W^S is finite).

Now we come to the crux of the matter: why can we define $\rho(t_1, m_1, t, m)$ for $(t_1, m_1) \in N_{\lambda(t,m)}$?

Let $(t_0, m_0) = (t_1, m_1) \restriction \lambda(t(0), m(0))$ and $\rho_2 = \rho(t_0, m_0, t(0), m(0))$, $l(\rho_2) = \mathrm{ht}(t(0), m(0))$, $\rho_2 \in t(0)$. Now we have to find $\rho, \rho_2 < \rho \in t \cap {}^{(n+1)}2$, *and* to satisfy condition (f) from (B) of the Main Definition 5.10.

So we are interested in the cases from (f) for which (ii) fail. Each one demands that $\rho(l) = 0$ for some $l < n+1$, and clearly $l \geqq l(\rho_2)$ in the cases which are not trivially satisfied.

How many such demands can we satisfy? Remember that as $(t(0), m(0)) \in M_{k,\leqq n}$, $m(0)(\rho_2) > 2^{-\mathrm{ht}(t(0),m(0))}(1 - 1/\mu(k))$, and as $(t, m) \in M_{k+1,n+1}$, $\{\nu \in {}^{(n+1)}2 : \nu \in t,\ \rho_2 \lhd \nu\}$ has $> 2^{(n+1)-l(\rho_2)}(1 - 1/\mu(k))$ members, so we can satisfy any $< \log_2 \mu(k)$ such demands.

Now, how many demands are there? We shall see for each pair l, j, for how many $\xi \in W^S$, $(t_1)_{[\eta_l]}$ is disjoint to $\bigcup_{\alpha<k+1} A^{(t,m),\xi}_{\alpha,j}$, in fact even to $A^{(t,m),\xi}_{k,j}$. Now we use the definition of the $A^{(t,m),\xi}_{k,j}$. Remembering that by the definition of N, η_l,

$$m_1(\eta_l) \geqq 1/4^{l(\eta_l)+1} \geqq 1/4^{l+1}$$

clearly

$$|(t_1)_{[\eta_l]}| / 2^{\lambda(t(0),m(0)) - l(\eta_l)} > 1/4^{l+1}.$$

If it is disjoint to x of the sets $\{A^{(t,m),\xi}_{k,j} : \xi \in W^S\}$, then by trivial probabilistic results (as $|\{\nu \in A^{(t,m),\xi}_{k,j} : \eta_l \lhd \nu\}| = |A^{(t,m),\xi}_{k,j}| / 2^{l(\eta_l)}$):

$$1/4^{l+1} < (1 - 1/2^{k+j+l})^x.$$

But $(1 - 1/2^{k+j+1})^{2^{k+j+l}} < 1/e < \frac{1}{2}$ (remember e is here the basis of the natural logarithms and $j > 0$), hence for $x > 2^{k+j+1}(2l+2)$ we get a contradiction. So $x < (2l+2)^{k+j+1}$.

So the number of pairs of such ξ, ζ is $< (2^{k+j+1}(2l+2))^2$, and we have to consider every $l < k$, $j < k$, $j > 0$. Hence the number of "problematic" l, j, ξ, ζ is at most

$$\sum_{l,j<k} (2^{k+j+1}(2l+2))^2 < 2^{5k+7}.$$

So if $\log_2(\mu(k)) > 2^{2^k}$, we finish, and this holds (see Definition 5.4 of $\mu(k)$).

Remark. We have been "generous" in our use of $A_{l,k}^{(l,m),\xi}$ and computations, hence of $\mu(k)$.

5.13. Lemma. *There is a full system in $L[a^*]$ (the only need for a^* is that $\aleph_1^{L[a^*]} = \aleph_1$).*

Proof. The existence of a full system is equivalent to the existence of some model for a sentence in $L_{\omega_1,\omega}(Q)$, hence is absolute (by Keisler completeness theorem), hence if we find such a system in a generic extension of $L[a^*]$, this is sufficient.

So just force with the family of finite system. By Lemma 5.12, $\aleph_1$ is not collapsed, and in a similar way to the proof of 5.12 we can show the required density demands.

Convention. Let S be such a system.

5.14. Definition. We define formulas (with real parameters) $\phi_{rd}(x) =$ *there are* perfect trees T_0, T_1 such that:

(a) T_0 is a poor man generic tree over $L[a^*]$ as in Claim 5.7, so for some $n(k)$ $(k < \omega)$, $(t(k), m(k)) = (T_0 \restriction {}^{n(k)>}2,\ ms_T \restriction {}^{n(k)>}2) \in M_k$.

(b) T_1 is a poor man generic tree over $L[a^*, T_0]$, T_1 as in Claim 5.6, so for every n, $(t_n, m_n) = (T_1 \restriction {}^{n\geq}2,\ ms_{T_1} \restriction {}^{n\geq}2) \in N_n$.

(c) For every k, $\rho_{rd}^S(t_{\lambda(t(k),m(k))}, m_{\lambda(t(k),m(k))}, t(k), m(k))$ is an initial segment of x.

(So the parameters which appear are ρ_{rd}^S and a^*, though we can eliminate ρ_{rd}^S by choosing a simply defined one.)

$\phi'_{rd}(x) =$ "there is y, $\phi_{rd}(y)$, and for all but finitely many $l < \omega$, $x(l) = y(l)$".

We define ϕ_{gr}, ϕ'_{gr} similarly.

5.15. Claim. The formulas above are Σ_3^1.

Proof. The non-trivial part is the "poor man genericity" which says: "for every B, which codes a G_δ-set of measure zero, B is not constructible from a^* *or* T is disjoint to B", which is $\mathbf{\Pi}^1_2$, as being contructible from a^* is $\mathbf{\Sigma}^1_2$.

5.16. CLAIM. The formulas $\phi'_{\mathrm{rd}}(x)$, $\phi'_{\mathrm{gr}}(x)$ are contradictory.

PROOF. Using h and x we define a colouring to $[\omega_1]^2 : x(h(\xi,\zeta))$.

If $\phi_{\mathrm{rd}}(x)$, there are T_0, T_1 exemplifying it, with $n(k)$ $(k<\omega)$. We define, for $j<\omega$, $\eta_l\in T$, $\alpha<\omega$,

$$A_{j,l,\alpha}=\{\xi<\aleph_1:(T_1)_{[\eta_l]}\text{ is disjoint to}$$

$$\bigcup_{\substack{l<\omega\\ k<\omega}} C_{l,j}^{(t(k),m(k)),\xi}\quad\text{and}\quad f^{(t(k),m(k)),\xi}_{(t_{\lambda(t(k),m(k))},\,m_{\lambda(t(k),m(k))})}(\eta_l,j)=\alpha$$

$$\text{for every large enough } k\}.$$

Then $A_{j,l,\alpha}$ is homogeneously red. So ω_1 is the union of countably many homogeneously red sets. Similarly for ϕ_{gr}, so clearly $\varphi_{\mathrm{rd}}(x)$, $\varphi_{\mathrm{gr}}(x)$ are contradictory.

But we have to deal with ϕ'_{rd}, ϕ'_{qr}. So suppose $\phi_{\mathrm{rd}}(x)$, $\phi_{\mathrm{gr}}(y)$, and say $\{(0,\dots,n^*\}\supseteq\{n:x(n)\neq y(n)\}$, is finite. So there is a homogeneously red set A for x and homogeneously green set B for y, $A\cap B$ uncountable. There is an infinite subset $\{\xi_n:n<\omega\}\subseteq A\cap B$ such that the truth value of $h(\xi_{n_1},\xi_{n_2})<h(\xi_{n_2},\xi_{n_3})$ is fixed (for $n_1<n_2<n_3$). By Definition 5.10, part (A), $h(\xi_n,\xi_{n+1})$ is strictly increasing, so for n large enough it is $>n^*$. So x "thinks" $\langle\xi_n,\xi_{n+1}\rangle$ is red, whereas y "thinks" it is green, but they agree. Contradiction.

REMARK. In fact $\phi'_{\mathrm{rd}}(x)$ implies x is a "red real" (by h), hence the contradiction.

5.17. CLAIM. $A_{\mathrm{rd}}=\{x:\phi'_{\mathrm{rd}}(x)\}$ is not of measure zero. Similarly for green.

PROOF. If b is a code of a G_δ set covering A_{rd}, which has measure zero, then by Claims 5.6, 5.7 there is T_0, a poor man generic tree over $L[a^*,b]$, and T_1, a poor man generic tree over $L[a^*,b,T_0]$. We can easily find x for which they are witnesses to $\phi'_{\mathrm{rd}}(x)$ (now $x\in\operatorname{Lim}T_0$), hence x is in no measure-zero set coded in $L[a^*,b]$, in particular the one b codes.

5.18. CLAIM. $\{x:\phi'_{\mathrm{rd}}(x)\}$ is not measurable.

PROOF. By 5.17, it is not of measure zero, but by its definition the measure of $\{x:\phi'_{\mathrm{rd}}(x),\ \eta\lhd x\}$ $(\eta\in{}^{\omega>}2)$ depends on $l(\eta)$ only. So by measure theory its outer measure is 1. But the same holds for $\{x:\phi'_{\mathrm{gr}}(x)\}$, and they are disjoint. Contradiction

PROOF OF THEOREM 5.1B. In the Main Definition 5.10, let W be an ordered set and let T be a closed subtree of ${}^{\omega>}2$ such that $|T \cap {}^n2| = n$, and we define by induction on n a finite system $S(n)$, such that:

(a) $W^{S(n)} = T \cap {}^n2$, ordered lexicographically.

(b) $\eta_n \in T \cap {}^n2$ is the unique η, $\eta\,\hat{}\,\langle 0\rangle$, $\xi\,\hat{}\,\langle 1\rangle \in T$ and $W^n(n,l) = W^{S(n+1)} - \{\eta_n\,\hat{}\,\langle 1-l\rangle\}$.

(c) The mapping $\eta \to \eta \restriction n$ is an isomorphism from $S(n+1) \restriction W(n,l)$ onto $S(n)$.

(d) $h^{S(n+1)}(\eta_n\,\hat{}\,\langle 0\rangle, \eta_n\,\hat{}\,\langle 1\rangle) = n$, $n(0)^{S(n+1)} = n+1$.

The induction step is just like the proof of 5.12, except for the definition of h which is handled by (d) (and trivially satisfies the last demand in 5.10(A)).

Now, if we have (in V) a set of $\aleph_1$ reals, then we have a sequence of length ω_1 of distinct members of Lim T, and restricting the inverse limit of the $S(n)$'s to this sequence, we get a system S, $|W^S| = \aleph_1$. However, the order used at the end of Definition 5.10(A) for the demand on h, is not a well-ordering. This demand was used only in the proof of 5.16, but the well-ordering was not used.

§6. On "every Δ^1_3 set of reals is measurable"

6.1. MAIN THEOREM. *Every universe V of set theory has a generic extension in which every Δ^1_3 set of reals is measurable.*

PROOF. We prove the theorem by the following series of claims.

6.2. DEFINITION.[†] Let $\varphi = \varphi(x)$, $\psi = \psi(x, y)$ be formulas and let $P_{\varphi,\psi}$ be the following forcing notion: the set of elements is $\{x : x \text{ a real and } \varphi(x)\}$, and the order is $x \leqq y$ iff $\psi(x, y)$. The formulas may have parameters.

Note that $P_{\varphi,\psi}$ depends on the universe, so we write explicitly $P_{\varphi,\psi}(V)$.

6.3. CLAIM. Suppose $V_1 \subseteq V_2$ are universes with the same ordinals, $\varphi(x)$ $\psi(x, y)$ are Σ^1_2, Σ^1_1, respectively (with parameters, if at all, from V_1), and $P_{\varphi,\psi}(V_1)$ is a forcing notion and satisfies the countable chain condition. Then:

(a) $P_{\varphi,\psi}(V_1) \subseteq P_{\varphi,\psi}(V_2)$ as ordered sets (and $P_{\varphi,\psi}(V_2)$ is a forcing notion).

(b) If $G \subseteq P_{\varphi,\psi}(V_2)$ is generic (over V_2) then $G \cap P_{\varphi,\psi}(V_1)$ is generic over V_1.

PROOF.

(a) Easy by the well-known absoluteness results.

[†] For more on Borel forcing see *On cardinal invariants of the continuum*, Proc. Conf. on Set Theory, Boulder, 1983.

(b) We just have to prove that any maximal antichain I of $P_{\varphi,\psi}(V_1)$ in V_1, is a maximal antichain in $P_{\varphi,\psi}(V_2)$. As $P_{\varphi,\psi}(V_1)$ satisfies the countable chain condition, I is countable, so let $I = \{x_n : n < \omega\}$ (maybe with repetitions). As it is a maximal antichain

$$V_1 \models (\forall y \in \mathbf{R})[\neg \varphi(y) \vee (\exists z)(\psi(y,z) \wedge \bigvee_n \psi(x_n, z))].$$

This is a $\mathbf{\Pi}^1_2$ statement, hence by the absoluteness theorem also V_2 satisfies it, hence I is a maximal antichain in V_2.

REMARK. Note that (b) implies that if $x, y \in P_{\varphi,\psi}(V_1)$ are compatible (incompatible), then the same holds in $P_{\varphi,\psi}(V_2)$.

6.4. CLAIM. Suppose for $i < \alpha$, φ_i, ψ_i are $\mathbf{\Sigma}^1_2$, $\mathbf{\Sigma}^1_1$ formulas defining in V a forcing notion. Let $\langle P_i, \underset{\sim}{P}_{\varphi_j,\psi_j} : i \leqq \alpha, j < \alpha \rangle$ be a finite support iteration, i.e.,

$$P_i = \{f : f \text{ a function with domain a finite subset of } i,$$
$$f(j) \text{ is a } P_j\text{-name of a condition in } P_{\varphi_j,\psi_j}(V^{P_j})\},$$
$$f \leqq g \text{ iff for every } j \in \operatorname{Dom} f,\ g \restriction j \Vdash_{P_j} \text{“}\psi_j(f(j), g(j))\text{”}.$$

Suppose further $\langle P^*_i, \underset{\sim}{P}_{\varphi^*_j,\psi^*_j} : i \leqq \beta, j < \beta \rangle$ is another such iteration, $\beta < \omega_1$, and there are $\gamma(i,n) < \alpha$ for $i < \beta$ such that $\gamma(i,n) < \gamma(i, n+1)$, $\gamma(i_1, n_1) < \gamma(i_2, n_2)$ for $i_1 < i_2$ and $\langle \varphi^*_j, \psi^*_j \rangle = \langle \varphi_{\gamma(j,n)}, \psi_{\gamma(j,n)} \rangle$. Assume also all P_{φ_j,ψ_j} are non-trivial, i.e., contain two incompatible elements.

Then we conclude that in V^{P_α} there is a subset of P^*_β generic over V; moreover, there are $\aleph_0$ such sets whose union is P^*_β.

REMARK. So we here assume $\langle \varphi_i, \varphi_i : i < \alpha \rangle \in V$.

PROOF. Let $Q_j = \underset{\sim}{P}_{\varphi_j,\psi_j}$, $Q^*_j = \underset{\sim}{P}_{\varphi^*_j,\psi^*_j}$ where $\langle \varphi^*_i, \psi^*_i : i < \beta \rangle$ and the function γ are as in the claim.

We prove by induction on $\zeta \leqq \beta$ that, letting $\xi(\zeta) = \bigcup\{\gamma(j,n) : j < \zeta, n < \omega\}$, in $V^{P_{\xi(\zeta)}}$ there is a subset of P^*_ζ generic over V, and even for every $\zeta_0 < \zeta$, $G^* \subseteq P^*_{\zeta_0}$ generic over V, $G^* \in V^{P_{\xi(\zeta_0)}}$ there are $G_n \subseteq P^*_\zeta$, generic over V, $G_n \cap P^*_{\zeta_0} = G^*$, $\bigcup_{n<\omega} G_n = P^*_\zeta / G = \{p \in P^*_\zeta : p \text{ compatible with every member of } G^*\}$ (and $\langle G_n : n < \omega \rangle \in V^{P_{\xi(\zeta)}}$).

$\zeta = 0$. Nothing to prove.

$\zeta + 1$. Clearly, it suffices to prove the statement for $\zeta_0 = \zeta$, $G^* \subseteq P^*_\zeta$ generic over V, $G^* \in V[G \cap P_{\xi(\zeta)}]$ where $G \subseteq P_{\xi(\zeta+1)}$ is generic over V.

For $k < \omega$ let $G_k \in V^{P_\xi}$ be the intersection of $P_{\varphi^*_\zeta,\psi^*_\zeta}(V[G^*])$ and the generic subset of $Q_{\gamma(\zeta,k)}(V^{P_{\gamma(\zeta,k)}})$ (this is done in $V[G]$).

It is easy to check that $\{G_k : k < \omega\}$ is as required.

ζ *Limit.* Let $\zeta = \bigcup_{n<\omega} \zeta_n$, $\zeta_n < \zeta_{n+1}$, ζ_0 the given one, and $G^* \subseteq P^*_{\zeta_0}$ generic over V, $G^*_1 \in V^{P_{\xi(\zeta_0)}}$. Let p_n, q_n be $P_{\gamma(\zeta_n,0)}$, two incompatible members of $Q_{\gamma(\zeta_n,0)}$.

We can define by induction on n, for every $\eta \in {}^n\omega$, $G_\eta \subseteq P^*_{\zeta_n}$ generic over V, $G_{\langle\,\rangle} = G^*$, $G_\eta \cap P^*_{\zeta_m} = G_{\eta\restriction m}$ for $m < n$ and $\bigcup_{i<\omega} G_{\eta^\frown\langle i\rangle} = P^*_{\zeta_{n+1}}/G_\eta$. Define $P_{\xi(\zeta)}$-names $\underset{\sim}{k}(n)$.

$\underset{\sim}{k}(0) = 0$, $\underset{\sim}{k}(n+1)$ is the first $k > \underset{\sim}{k}(n)$ such that p_n is in the generic subset of $Q_{\gamma(\zeta_n,k)}$. As the support is finite, and by the definition of generic, it is easy to check the $\bigcup\{\bigcup_{l<\omega} G_{\eta\restriction l} : \eta \in {}^\omega\omega$ is in $V^{P_{\xi(\zeta)}}$ and for every large enough n, $\eta(n) = \underset{\sim}{k}(n)\}$ is as required.

Proof of the Theorem. Let $\langle P_i, \underset{\sim}{Q}_j : i < \omega_1, j < \omega_1\rangle$ be an iterated forcing with finite support, Q_j is the random real forcing for α even and the amoeba forcing Am for α odd (AM $= \{A \subseteq [0,1] : A$ an open set of measure $< \frac{1}{2}\}$, ordered by inclusion). Let $G_{\omega_1} \subseteq P_{\omega_1}$ be generic.

So let $\varphi(x,r)$ be a $\mathbf{\Delta}^1_3$ formula, r a real parameter, and it suffices to prove that for some A of positive measure, $\{x : \varphi(x,r)\} \cap A$ is measurable.[1] As is well known, each Q_i satisfies the countable chain condition, hence also P_{ω_1} satisfies it, so for some $\alpha < \omega_1$, $r \in V^{P_\alpha}$, so w.l.o.g. $\alpha = 0$. By symmetry, suppose that the random real r^* of Q_0 (i.e., the one we get from $G_{\omega_1} \cap P_1$) satisfies $\varphi(x,r)$. As $\varphi(x,r)$ is $\mathbf{\Delta}^1_3$ for some $\mathbf{\Pi}^1_2$ formula $\psi(x,y,z)$, $\varphi(x,r) = \exists y\, \psi(x,y,r)$, hence form some $r_1 \in V[G_{\omega_1}]$, $V[G_{\omega_1}] \models \psi(r^*, r_1, r)$, but again for some $\beta < \omega_1$, $r_1 \in V[G_\beta]$. By the absoluteness lemma also $V[G_\beta] \models \psi[r^*, r_1, r]$. Hence for some $p \in P_\beta$, $p \Vdash_{P_\beta} \psi(\underset{\sim}{r}^*, \underset{\sim}{r}_1, r)$, $\underset{\sim}{r}_1$ a P_β-name, $\underset{\sim}{r}^*$ the P_1-name of the random real. Note that $P_1 = Q_0$.

Now clearly there is $q_0 \in Q_0$, $q_0 \Vdash$ "$p \in P_\beta / Q_0$". Now

(*) every real $r^+ \in V[G_{\omega_1}]$ which is random over V and belongs to q_0 (i.e., r^+ defines a subset of Q_0 generic over V which includes q_0) satisfies φ.

This is clearly sufficient, as the amoeba forcing makes the union of all Borel sets with old codes of measure zero, into a set of measure zero, so except for measure-zero sets, every real in $V[G_{\omega_1}]$ is random over V.

So let us prove (*).

But by the previous claim, and what we prove in its proof, for some $G \in V[G_{\omega_1}]$, $V[r^+, G] \models \psi[r^+, \underset{\sim}{r}_1[G], r]$, hence $V[r^+, G] \models \varphi[r^+, r]$, hence by absoluteness $V[G_{\omega_1}] \models \varphi[r^+, r]$.

Concluding Remarks.

(1) We can get more information on forcings with "simple" definitions.

[1] As we are proving this for every $\varphi(x,r)$.

(2) The proof clearly has very little to do with random reals.

We can notice

6.5. LEMMA. (1) *Suppose* $\langle P_i, \underset{\sim}{P}_{\varphi_j,\psi_j} : i \leqq \alpha, j < \alpha \rangle$ *is a finite support iteration, where* φ_j, ψ_j *are* Σ^1_2, Σ^1_1 *formulas, respectively, and* $\underset{\sim}{P}_{\varphi_j,\psi_j}$ $(\in V^{P_j})$ *satisfies the* $\aleph_1$-cc.

If w *is a subset of* α, $\beta = \bigcup_{i \in w}(i+1)$, $\langle P'_i, \underset{\sim}{P}_{\varphi_j,\psi_j} : i \in w \bigcup \{\beta\}, j \in w \rangle$ *is also a finite support iteration, then* $P'_\beta \lessdot P_\alpha$.

PROOF. We prove this lemma by induction on β. For $\beta = 0$ this is trivial, for β successor use Claim 6.3, and for β limit there are no special problems.

§7. On "every set of reals has the Baire property"

7.1. NOTATION AND STANDARD FACTS. Let P, Q, R denote forcing notions, i.e., quasi-orders with minimal element ϕ so $x \leqq y \wedge y \leqq x$ may not imply $x = y$. Let their members be denoted by p, q, r. We denote complete Boolean algebras by B but reverse their order (so $1_B < 0_B$) and omit the zero element. It is well known that for every P, there is a complete Boolean algebra $B = \mathrm{BA}(P)$ and $f : P \to B$ such that $p \leqq q \Rightarrow f(p) \leqq f(q)$, p, q incompatible iff $f(p) \cap f(q) = 0$, and $\{f(p) : p \in P\}$ is dense in $B - \{0\}$, i.e., $(\forall b \in B)$ $(\exists p \in B)$ $[b > f(p)]$. Moreover, (f, B) are unique up to isomorphism over P, and there is no difference between forcing by P and forcing by B.

We say $I \subseteq P$ is dense [above p] if for every $q \in P$ (such that $q \geqq p$) there is $r \in I$, $r \geqq q$.

We say P, Q are equivalent if $\mathrm{BA}(P) \cong \mathrm{BA}(Q)$

We say $P < Q$ if

(a) for p, $q \in P$, $p \leqq q$ in P iff $p \leqq q$ in Q,

(b) for p, $q \in P$, p, q are compatible in P iff they are compatible in Q,

(c) every maximal antichain of P is a maximal antichain of Q.

So $P < \mathrm{BA}(P)$ (if we identify p and $f(p)$).

Let $\underset{\sim}{G}_P$ be the P-name of the generic subset of P.

If $P < Q$ or even $P < \mathrm{BA}(Q)$ let $Q/P = \{q \in Q : q$ is compatible with every $p \in \underset{\sim}{G}_P\}$ (so this is a P-name of a forcing notion, which is a subset of Q).

Note $p \Vdash$ "$q \in Q/P$" iff every $p' \in P$, $p' \geqq p$ is compatible with q.

It is well known that Q and $P * (Q/P)$ are equivalent when $P < \mathrm{BA}(Q)$.

If $P_0 < \mathrm{BA}(P_1)$, $\mathrm{BA}(P_2)$, let

$$P_1 *_{P_0} P_2 = \{(p_1, p_2) : p_e \in P_e \text{ for } e = 1,2 \text{ and for some } p_0 \in P_0,\ p_0 \Vdash \text{“} p_e \in P_e / P_0 \text{”} \text{ for } e = 1,2\}$$

with the natural order $(p_1, p_2) \leqq (p'_1, p'_2)$ iff $p_1 \leqq p'_1$, $p_2 \leqq p'_2$.

It is well known that $P_1 *_{P_0} P_2$, $P_0 * (P_1/P_0 \times P_2/P_0)$ are equivalent. Also, for $e = 1, 2$, $P_e < P_1 *_{P_0} P_2$ identifying p_1, (p_0, ϕ) and p_2, (ϕ, p_2), resp.

7.2. THE MAIN DEFINITION.

(1) The forcing notion P is *sweet* if there is a subset $\mathscr{D}$ of P, and equivalence relations E_n on $\mathscr{D}$, such that:

(a) E_{n+1} refines E_n, E_n has countably many equivalence classes, $\mathscr{D} \subseteq P$ is dense.

(b) For every $n < \omega$, $p \in \mathscr{D}$, p/E_n is dircted.

(c) If $p_i \in \mathscr{D}$ for $i \leqq \omega$, and $p_n E_n p_\omega$ then $\{p_i : i \leqq \omega\}$ has an upper bound; moreover, for each $n < \omega$, $\{p_i : n \leqq i \leqq \omega\}$ has an upper bound in p/E_n.

(d) For every p, q in $\mathscr{D}$ and $n < \omega$ there is $k < \omega$ such that for every $p' \in p/E_k$, $(\exists r \in q/E_n)$ $(r \geqq p)$ implies $(\exists r \in q/E_n)$ $(r \geqq p')$.

REMARK. Those statements (in (d)) are equivalent to:

q, $p(q, p')$ are compatible in q/E_n, i.e., have a common upper bound there; remember q/E_n is directed. If $p < q$, clearly for every $p' \in p/E_k$ there is $q' \in q/E_n$, $q' \geqq q$, $q' \geqq p'$ (really, if (d) holds for every $p < q$ (and (b)) and (a) holds).

7.2A. DEFINITION. We say that $\mathscr{D}$, E_n $(n < \omega)$ exemplify the sweetness of P if (1) holds, and call $(p, \mathscr{D}, E_n)$ a sweetness model.

7.3. CLAIMS.

(1) If $P < \mathrm{BA}(Q)$, Q is sweet, then P is the union of countably many subsets A_n, the elements of each A_n are pairwise compatible (in fact, P is the union of countably many directed subsets).

(2) We can replace $\mathscr{D}$ by any of its dense open subsets.

PROOF.

(1) Trivial. If $\mathscr{D}$, E_n exemplifies the sweetness of Q, then for $q \in \mathscr{D}$ let $A_q = \{p$: for some $q' \in q/E_0$, $p \leqq q'\}$; A_q is directed, there are $\aleph_0$ A_q's and $P = \bigcup_{q \in \mathscr{D}} A_q$.

(2) Trivial too. If $\mathscr{D}$, E_n $(n < \omega)$ exemplify the sweetness of P, $\mathscr{D}' \subseteq \mathscr{D}$ is dense, $p \in \mathscr{D}'$, $q \geqq p$, $q \in \mathscr{D}$ implies $q \in \mathscr{D}'$, then $\mathscr{D}'$, $E_n \restriction \mathscr{D}'$ $(n < \omega)$ exemplify the sweetness of P too.

7.4. CLAIM. Suppose $P < \mathrm{BA}(Q)$, Q is sweet and this is exemplified by $\mathscr{D}$, E_n $(n < \omega)$. Suppose further $A_n \subseteq P$ and $\bigcup_{n<\omega} A_n$ is a dense subset of P. *Then for any* $q \in \mathscr{D}$, and $p \in P$ such that $p \Vdash$ "$q \in Q/P$":

(1) For some n, $k < \omega$ the following holds:

(*) For any $q' \in q/E_k$, for some $p' \in A_n$, $p' \geqq p$ and $p' \Vdash_P$ "$q' \in Q/P$".

(2) Moreover $A^* = \bigcup\{A_n : A_n$ satisfies $(*)$ for some $k\}$ is dense in P above p.

PROOF.

(1) Let $\{n(i): i < \omega\}$ be a list of the natural numbers, each appearing infinitely many times. Define by induction on $i < \omega$, $q_i \in q/E_i$, such that:

$\bigoplus$ *If* there is $q' \in q/E_i$ such that for no $p' \in A_{n(i)}$, $p' \geqq p$ and $p' \Vdash_P$ "$q' \in Q/P$" *then* q_i has this property, i.e., for no $p' \in A_{n(i)}$, $p' \geqq p$ and $p' \Vdash_P$ "$q_i \in Q/P$".

There is no problem in the definition and as $\mathcal{D}$, E_n exemplify the sweetness of P there is $k < \omega$ such that every $q' \in q/E_k$ is compatible with p (choose $r \in \mathcal{D}$, $r \geqq p$, q, apply Definition 7.2(1)(d) for q, r). By Definition 7.2(1)(c) there is $q^* \in D$, $q^* \geqq q$, q_i for $k \leqq i < \omega$. Moreover $q^* \in q/E_k$. So by the choice of k, q^*, p are compatible, hence for some $p' \geqq p$ in P, $p' \Vdash_P$ "$q^* \in Q/P$". As $\bigcup_{n<\omega} A_n$ is dense in P, w.l.o.g. for some n, $p' \in A_n$.

By the choice of the $n(i)$ $(i < \omega)$, clearly, for some i, $k < i < \omega$, $n(i) = n$. As $q_i \leqq q^*$, clearly $p' \Vdash_P$ "$q_i \in Q/P$", hence in $\bigoplus$ (as $p' \in A_{n(i)}$) the conclusion fails, hence the assumption fails, i.e., for any $q' \in q/E_i$ for some $p'' \in A_{n(i)}$, $p'' \geqq p$ and $p'' \Vdash_P$ "$q' \in Q/P$". But this is the desired conclusion of (1).

(2) We can replace p by any $p_0 \geqq p$ $(p_0 \in P)$ in (1), so for every $p_0 \geqq p$ there is n satisfying $(*)$ (with p_0 instead of p), for some k_n. Apply $(*)$ for $q' = q$ and get $p' \geqq p_0$, $p' \in A_n$, and of course $A_n \subseteq A^*$.

7.5. THE AMALGAMATION LEMMA. *Suppose P_1, P_2 are sweet, $P_0 < \mathrm{BA}(P_1)$, $\mathrm{BA}(P_2)$, then $Q = P_1 *_{P_0} P_2$ is sweet.*

PROOF. Let, for $e = 1,2$, the sweetness of P_e be exemplified by $\mathcal{D}_e$, E^e_n $(n < \omega)$. By Claim 7.3(1) there are sets $A_n \subseteq P_0$, of pairwise compatible elements, and let $\mathcal{D} = \{(p_1, p_2): p_1 \in D_1, p_2 \in D_2, (p_1, p_2) \in Q\}$.

Suppose $(p_1, p_2) \in \mathcal{D}$, and let $p_0 \in P_0$ exemplify it, i.e., $p_0 \Vdash_{P_0}$ "$p_e \in P_e/P_0$" for $e = 1,2$. By Claim 7.4(2), $\bigcup_{n \in S} A_n \subseteq P_0$ is dense over p_0 where $S = \{n :$ for some $k_n < \omega$ for every $p'_1 \in p_1/E^1_{k_n}$, there is $p'_0 \in A_n$, $p'_0 \geqq p_0$, and $p'_0 \Vdash_{P_0}$ "$p'_1 \in P_0/P_1$"$\}$.

Apply again 7.4(2), this time for $P_0 < \mathrm{BA}(P_2)$, and $\{A_n : n \in S\}$, and we get that $\bigcup_{n \in W} A_n \subseteq P_0$ is dense over p_0 where $W = \{n \in S :$ for some $e_n < \omega$ for every $p'_2 \in p_2/E^2_{e_n}$ there is $p'_0 \in A_n$, $p'_0 \geqq p_0$ and $p'_0 \Vdash_{P_0}$ "$p'_2 \in P_2/P_0$"$\}$. Choose $n \in W$. Let $m = \mathrm{Max}\{e_n, k_n\}$, then for any $p'_1 \in p_1/E^1_m$, $p'_2 \in p_2/E^2_m$, there are p^1_0, $p^2_0 \geqq p_0$, p^1_0, $p^2_0 \in A_n$ and $p^e_0 \Vdash_{P_0}$ "$p'_e \in P_e/P_0$" for $e = 1,2$. But as p^1_0, $p^2_0 \in A_m$ they are compatible, and their common upper bound exemplify $(p'_1, p'_2) \in Q$. We have proved

(*) For every $(p_1, p_2) \in \mathscr{D}$ there is $m < \omega$ such that for every $p_1' \in p_1/E^1_m$, $p_2' \in p_2/E^2_m$,
there is $p_0 \in P_0$, $p_0 \Vdash_{P_0}$ "$p_i' \in P_1/P_0$" and $p_0 \Vdash_{P_0}$ "$p_2' \in P_2/P_0$".

Define for $(p_1, p_2) \in Q$, $m(p_1, p_2)$ as the minimal $m < \omega$ for which the statement (*) holds.

Now define the equivalence relations E_n on $\mathscr{D}$: $(p_1, p_2)\ E_l(p_1', p_2')$ iff $p_1 E^1_{m+l} p_1'$, $p_2 E^2_{m+l} p_2'$ where $m = m(p_1, p_2) = m(p_1', p_2')$. The checking of Definition 7.2(1) is trivial, e.g.,

Condition (a): Trivially E_e is symmetric and reflexive, and it is transitive. For suppose $(p_1^k, p_2^k) E_e (p_1^{k+1}, p_2^{k+1})$, for $k = 1, 2$, then $m(p_1^1, p_2^1) = m(p_1^2, p_2^2)$ and $m(p_1^2, p_2^2) = m(p_1^3, p_2^3)$ hence $m(p_1^1, p_2^1) = m(p_1^3, p_2^3)$, and call the common value m. So also $p_1^k E^1_{m+e} p_1^{k+1}$, $p_2^k E^2_{m+e} p_2^{k+1}$, for $k = 1, 2$, and as E^2_{m+e}, E^1_{m+e} equivalence relations, $p_1^1 E^1_{m+e} p_1^3$, $p_2^1 E^2_{m+e} p_1^3$. We can conclude $(p_1^1, p_2^1) E_e (p_1^3, p_2^3)$, so E_e is transitive.

It is quite clear that E_{e+1} refines E_e and E_e has

$$\leqq |\{p_1/E_n : p_1 \in \mathscr{D}_1, n < \omega\}| \times |\{p_2/E_n : p_2 \in \mathscr{D}_2, n < \omega\}| \leqq \aleph_0$$

equivalence classes.

Note also

(**) If $m = m(p_1, p_2), (p_1, p_2) \in \mathscr{D}$, and $p_1' \in p_1/E^1_m, p_2' \in p_2/E^2_m$,
then $m(p_1', p_2') = m(p_1, p_2)$.

7.6. The Composition Lemma. *If P is sweet and $\underset{\sim}{Q}$ (in V^P) is* UM *then $P * \underset{\sim}{Q}$ is sweet where*

7.7. Definition.

UM $= \{(t, T) : T \subseteq {}^{\omega>}2$ a perfect nowhere dense tree, $t = T \cap {}^{n>}2$ for some $n\}$,

$(t_1, T_1) \leqq (t_2, T_2)$ iff $T_1 \subseteq T_2$, $t_1 \subseteq t_2$ and $t_1 = t_2 \restriction n$ for some n.

Proof. Let $\mathscr{D}^0$, E^0_n $(n < \omega)$ exemplify the sweetness of P, and let $\{A_e : e < \omega\}$ enumerate $\{p/E^0_n : n < \omega, p \in \mathscr{D}^0\}$. Now define

$$\mathscr{D} = \{(p, (t, \underset{\sim}{T})) : p \in \mathscr{D}^0, \emptyset \Vdash_P \text{“}(t, \underset{\sim}{T}) \in \underset{\sim}{Q}\text{”}\}.$$

Clearly $\mathscr{D}$ is a dense subset of $P * \underset{\sim}{Q}$. Now we come to the main point, the definition of the E_n:

For $(p_l, (t_l, \underset{\sim}{T}_l)) \in \mathscr{D}$ $(l = 1, 2)$, $(p_1, (t_1, \underset{\sim}{T}_1)) E_n (p_2, (t_2, \underset{\sim}{T}_2))$ iff the following conditions hold:

(α) $p_1 E^0_n p_2$,

(β) $t_1 = t_2$,

(γ) for every $m < n$, there is $p \in A_m$, $p \geqq p_1$ iff there is $p \in A_m$, $p \geqq p_2$,

(δ) suppose $m < n$ and there is $p \in A_m$, $p \geqq p_1$ and let $\eta \in {}^{n>}2$, then there is $p \in A_m$, $p \Vdash_P$ "$\eta \not\in \underset{\sim}{T}_1$" iff there is $p \in A_m$, $p \Vdash_P$ "$\eta \not\in \underset{\sim}{T}_2$",

(ε) let $k_m(q)$ be the minimal k such that for every $q' \in q/E^0_k$, $(\exists r \in A_m)$ $(r \geqq q)$ implies $(\exists r \in A_m)(r \geqq q')$.

Now we demand that for $m < n$, $k_m(p_1) = k_m(p_2)$ and $p_1 E^0_{k_m(p_1)} p_2$ (but note that if (γ) holds, $p_1 E^0_{k_m(p_1)} p_2$ implies $k_m(p_1) = k_m(p_2)$.

The following fact is important:

7.8. Subclaim. Let $p \in \mathscr{D}^0$, $k = \operatorname{Max}\{k_m(p) : m < n\} \cup \{n\}$, and suppose $p' \in \mathscr{D}$, $p' E_k p$ and $p' \geqq p$. Then:

(1) For any $(t, \underset{\sim}{T})$ such that $(p, (t, \underset{\sim}{T})) \in \mathscr{D}$ also $(p', (t, \underset{\sim}{T})) \in \mathscr{D}$ and $(p, (t, \underset{\sim}{T})) E_n (p', (t, \underset{\sim}{T}))$.

(2) For any $m < n$, there is $q \in A_m$, $q \geqq p$ if and only if there is $q \in A_m$, $q \geqq p'$.

Proof of the Subclaim.

(1) The only problematic point is (γ) in the definition of E_n, which is just (2) of the Subclaim.

(2) Now the "if" part is obvious as $p' \geqq p$, and the "only if" part by the definitions of $k_m(p)$ and k.

Continuation of the Proof of 7.6. So we have to check the conditions in Definition 7.2(1).

Condition (a). Trivial (in proving that E_n is an equivalence relation, for (δ) note (γ) implies we can replace $p \geqq p_1$ by $p \geqq p_2$, and by $p \geqq p_1, p_2$; for (ε) note (δ) and that A_m is directed).

Condition (b). Let $(p_1, (t_1, \underset{\sim}{T}_n))$, $(p_2, (t_2, \underset{\sim}{T}_2))$ be E_n-equivalent, and we have to find a common upper bound equivalent to them. Let $k_m(p_i)$ be as in (ε). So by (ε), for $m < n$, $k_m(p_1) = k_m(p_2)$ for $m < n$ and let $k = \operatorname{Max}\{k_m(p_1) : m < n\} \cup \{n\}$. So by the definition of E_n, $p_1 E^0_k p_2$.

So as $\mathscr{D}^0$, E^0_n exemplifies the sweetness of P, and Definition 7.2(1)(b), clearly there is $p^* \geqq p_1, p_2$, $p^* E^0_k p_1$. A common bound is $(p^*, (t, \underset{\sim}{T}_1 \cup \underset{\sim}{T}_2))$ where $t = t_1 = t_2$ (by (β)). Easily it belongs to $P * \underset{\sim}{Q}$ and even to $\mathscr{D}$.

Let us check the conditions for $(p_1, (t_1, \underset{\sim}{T}_1)) E_n (p^*, (t, \underset{\sim}{T}_1 \cup \underset{\sim}{T}_2))$.

(α) As $k \geqq n$, E^0_k refines E^0_n and $p^* E^0_k p_1$, clearly $p^* E^0_n p_1$.

(β) Holds by the choice of t.

(γ) Holds as $p^* E^0_k p_1$, and by the choice of k and Subclaim 7.8(2).

(δ) Suppose $m < n$, $\eta \in {}^{n>}2$ and there is $p \in A_m$, $p \geqq p_1$, so as A_m is directed and by (γ) there is such $p \geqq p_1, p_2, p^*$. By (δ) for $(p_1, t, \underset{\sim}{T}_1)E_n(p_2, (t, \underset{\sim}{T}_2))$; there is $r \in A_m$, $r \Vdash_P$ "$\eta \notin \underset{\sim}{T}_1$" if and only if there is $r \in A_m$, $r \Vdash_P$ "$\eta \in \underset{\sim}{T}_2$". So if there is $r \in A_m$, $r \Vdash_P$ "$\eta \notin \underset{\sim}{T}_1$", then, as A_m is directed, there is $r \in A_m$, $r \Vdash_P$ "$\eta \notin \underset{\sim}{T}_1 \cup \underset{\sim}{T}_2$". The inverse is even easier: if there is $r \in A_m$, $r \Vdash_P$ "$\eta \notin \underset{\sim}{T}_1 \cup \underset{\sim}{T}_2$", clearly for the same $r \in A_m$, $r \Vdash_P$ "$\eta \notin \underset{\sim}{T}_1$".

(ε) Trivial. (Use 7.8 again.)

So we have proved $(p^*, (t, \underset{\sim}{T}_1 \cup \underset{\sim}{T}_2))E_n(p_1, (t, \underset{\sim}{T}_1))$ hence prove condition (b) for Definition 2.1.

Condition (c). Suppose $(p_i, (t_i, \underset{\sim}{T}_i))E_i(p_\omega, (t_\omega, \underset{\sim}{T}_\omega))$ for $n \leqq i < \omega$ and we have to prove that the set $\{(p_i, (t_i, \underset{\sim}{T}_i)) : n \leqq i \leqq \omega\}$ has an upper bound in $(p_\omega, (t_\omega, \underset{\sim}{T}_\omega))/E_n$. We know that for $i < \omega$, $i \geqq n$, $p_i E_k p_\omega$ where $k = \operatorname{Max}\{k_m(p_\omega) : m < n\} \cup \{n\}$. So by condition (c) of Definition 7.2(1) for P, there is p^*, $p^* \geqq p_i$ for $n \leqq i \leqq \omega$, and $p^* \in p_\omega / E^0_k$ (really a short argument is needed, but such an argument appears in the proof of $\emptyset \Vdash$ "$\underset{\sim}{T}^*$ is nowhere dense"). By (β) we have $t_i = t$ for every i. We shall prove that $(p^*, (t, \underset{\sim}{T}^*))$ is an upper bound as desired where

$$\underset{\sim}{T}^* = \begin{cases} \bigcup\limits_{n \leq i \leq \omega} \underset{\sim}{T}_i & \text{if } p^* \in \underset{\sim}{G}_P, \\ \underset{\sim}{T}_\omega & \text{otherwise.} \end{cases}$$

Let us check that

$$\underset{\sim}{T}^* \textit{ a perfect subtree of } {}^{\omega>}2, \emptyset \Vdash \text{“}\underset{\sim}{T}^* \cap {}^{l>}2 = t \textit{ for the suitable } l\text{”}$$

is trivial.

The main point is

$$\emptyset \Vdash \text{“}\underset{\sim}{T}^* \textit{ is nowhere dense}\text{”}.$$

So let $\eta \in {}^{\omega>}2$, $r \in P$, and we have to find $p' \geqq r$, $\nu \in {}^{\omega>}2$, $\eta \triangleleft \nu$ such that $p' \Vdash_P$ "$\nu \notin \underset{\sim}{T}^*$"; w.l.o.g. $r \geqq p^*$ or r, p^* are incompatible. In the second possibility the statement is trivial so assume $r \geqq p^*$. As $(p_\omega, (t, \underset{\sim}{T}_\omega)) \in \mathscr{D}$, there is $r' \geqq r$ such that for some $\nu \in {}^{\omega>}2$, $\eta \triangleleft \nu$, $r' \Vdash_P$ "$\nu \notin \underset{\sim}{T}_\omega$", and w.l.o.g. $r' \in \mathscr{D}^0$. For every i, $n \leqq i < \omega$, $i > l(\nu)$ let $l(i) < \omega$ be maximal such that for some $m(i) < i$, $r'/E^0_{l(i)} = A_{m(i)}$. Suppose $l(i)$ exists, which occurs for every i large enough, say $> i_0$. So for every j, $i_0 \leqq j \leqq \omega$, as $(p_\omega (t, \underset{\sim}{T}_\omega))E_i(p_j, (t, \underset{\sim}{T}_j))$ and as $m(j) < j$, and as there is $p \in A_{m(j)}$, $p \geqq p_\omega$ (take $p = r'$), by (γ) of the definition of E_i there is

$r_j^0 \in A_{m(j)}$, $r_j^0 \geqq p_j$. Similarly by (δ) there is $r_j^1 \in A_{m(j)}$, $r_j^1 \Vdash_P$ "$\nu \notin \underset{\sim}{T}_j$" (remember $l(\nu) < i_0 \leqq j$). Let $r_j \in A_{m(j)}$, $r_j \geqq r_j^1$, r_j^0 (exists as $A_{m(j)}$ is directed).

Notice that $r_i E^0_{l(i)} r'$, and $l(i)$ may be $< i$, but it diverges to infinity, and $l(i) \leqq l(i+1)$. Let i_k be the first i such that $l(i) > k$, so $\{r_{i_k}, \ldots, r_{i_{k+1}-1}\}$ are all E^0_k-equivalent to r', so they have a common bound $r'_k \in r'/E^0_k$; if $i_k = i_{k+1}$, let $r'_k = r'$. Hence the set $\{r'_k : n \leqq k < \omega\} \cup (r')$ has a bound r^*, so $r^* \geqq r'$, r_i for $i \geqq i_n$, and clearly $r^* \Vdash_P$ "$\nu \notin \underset{\sim}{T}_i$" for i, $l(n) \leqq i < \omega$ and also for $i = \omega$ (as $r^* \geqq r'$). However $\emptyset \Vdash$ "$\underset{\sim}{T}_i$ is nowhere dense" for every i. So we can define by induction on i, $n \leqq i > i_n + 1$, r_i^*, $\nu_i \in {}^{\omega >}2$, such that $r_n^* = r^*$, $\nu_n = \nu$, $\nu_i \lessdot \nu_{i+1}$, and $r^*_{i+1} \Vdash$ "$v_i \notin \underset{\sim}{T}_i$". Now $r^*_{i_n+1}$, ν_{i_n+1} are as required, as $r^*_{i_n+1} \Vdash$ "$\nu_{i_n+1} \notin \underset{\sim}{T}_i$" for every i, $n \leqq i \leqq \omega$ (for $i > i_n$ by the choice of r^* and r'_i, for $n \leqq i \leqq i_n$ by the choice of $r^*_{i+1} \leqq r^*_{i_n+1}$). So $r^*_{i_n+1} \Vdash$ "$\underset{\sim}{T}^*$ is nowhere dense", and we can conclude that $(p^*, (t, \underset{\sim}{T}^*)) \in \mathcal{D}$. Now

$$(p^*, (t, \underset{\sim}{T}^*)) \geqq (p_i(t, \underset{\sim}{T}_i)) \qquad \text{for } n \leqq i \leqq \omega$$

trivially, and we shall prove

$$(p^*, (t, T^*)) E_n (p_\omega, (t, \underset{\sim}{T}_\omega)).$$

We have to check (α)–(ε). This is similar to the proof of condition (b), but nevertheless we check.

As $p^* E^0_k p_\omega$, $p^* \geqq p_\omega$, $k = \operatorname{Max}\{k_n(p_\omega) : m < n\} \cup \{n\}$ it is easy to check (α), (γ), (ε). Now (β) is trivial, so we remain with (δ). Let $m < n$ and assume there is $r \in A_m$, $r \geqq p_\omega$ (or equivalently $r \geqq p^*$). If $\eta \in {}^{n>}2$, and there is $p \in A_m$, $p \Vdash_P$ "$\eta \notin \underset{\sim}{T}^*$" then trivially for the same $p \in A_m$, $p \Vdash_P$ "$\eta \notin \underset{\sim}{T}_\omega$". Suppose $p \in A_m$, $p \Vdash_P$ "$\eta \notin \underset{\sim}{T}_\omega$", $p \geqq p_\omega$. So, as in the previous argument, there is $p' \geqq p$, $p' \in A_m$ and for some k, $p' \Vdash$ "$\eta \notin T_i$" for $k \leqq i \leqq \omega$. As $(p_i, (t, \underset{\sim}{T}_i)) E_n (p_\omega, (t, \underset{\sim}{T}_\omega))$ there are $p'_i \in A_m$ $(n \leqq i < k)$, $p'_i \Vdash$ "$\eta \notin \underset{\sim}{T}_i$". So as A_m is directed $\{p'_i : n \leqq i < k\} \cup \{p'\}$ has an upper bound in A_m, p'', and clearly $p'' \Vdash_P$ "$\eta \notin \underset{\sim}{T}^*$".

Condition (d). Suppose $(p, (t, \underset{\sim}{T})) \leqq (q, (s, \underset{\sim}{S}))$ are in $\mathcal{D}$ and $n < \omega$. Let $k_1 = \operatorname{Max}\{k_m(q) : m < n\} \cup \{n\}$, and let $k_2 > k_1$ be such that for every $p' \in p/E^0_{k_2}$, p', q have an upper bound in $q/E^0_{k_1}$. Let $q/E^0_{k_1}$ be $A_{m(0)}$ and $n(0)$ be minimal such that $s \subseteq {}^{n(0)>}2$ and let $k = k_2 + m(0) + n(0) + 1$.

We claim that $(p', (t, T')) E_k (p, (t, \underset{\sim}{T}))$ implies that $(p', (t, T'))$, $(q, (s, \underset{\sim}{S}))$ have a common upper bound in $(q, (s, \underset{\sim}{S}))/E_n$.

We know by k_2's definition that there is $p_0^* \geqq p'$, p, q, such that $p_0^* E^0_{k_1} q$ (as $k \geqq k_2$, $p' E^0_k p$). Because $(p, (t, \underset{\sim}{T})) \leqq (q, (s, \underset{\sim}{S}))$, clearly for $\eta \in {}^{n(0)>}2 - s$, $q \Vdash$ "$\eta \notin T$" so $(\exists r \in A_{m(0)})(r \geqq p \wedge r \Vdash$ "$\eta \notin T$"), hence as $(p', (t, T')) E_k$

$(p,(t,\underset{\sim}{T}))$, $m(0)$, $n(0)<k$ there is $p^*_\eta\in A_{m(0)}$, $p^*_\eta\Vdash$ "$\eta\notin\underset{\sim}{T}'$". Let $p^*\in q/E^0_{k_1}=A_{m(0)}$ be a common upper bound of $\{p^*_0,p^*_\eta:\eta\in{}^{n(0)>}2,\ \eta\in s\}$.

Let

$$\underset{\sim}{T}^*=\begin{cases}\underset{\sim}{S}\cup\underset{\sim}{T}' & \text{if } p^* \text{ is in the generic subset } \underset{\sim}{G}_P,\\ \underset{\sim}{S} & \text{otherwise.}\end{cases}$$

Clearly if $(p^*,(s,\underset{\sim}{T}^*))\in\mathscr{D}$, then it is $\geqq(p',(t,\underset{\sim}{T}'))$, and $\geqq(q,(s,\underset{\sim}{S}))$.

First let us prove $(p^*,(s,\underset{\sim}{T}^*))\in\mathscr{D}$, clearly $p^*\in\mathscr{D}^0$, so we have to prove only $\Vdash_P$ "$(s,\underset{\sim}{T}^*)\in\underset{\sim}{UM}$". Clearly $\Vdash_P$ "$\underset{\sim}{T}^*$ is a perfect nowhere-dense tree", so we have to prove for $\eta\in{}^{n(0)>}2$, $\Vdash$ "$\eta\in\underset{\sim}{T}^*$ iff $\eta\in s$". If $\eta\in s$, then because $(q,(s,\underset{\sim}{S}))$ is a condition $\Vdash_P$ "$\eta\in\underset{\sim}{S}$" but $\Vdash_P$ "$\underset{\sim}{S}\subseteq\underset{\sim}{T}^*$", hence $\Vdash_P$ "$\eta\in\underset{\sim}{T}^*$". For the other direction assume $\eta\in{}^{n(0)>}2-s$, $r\in P$. W.l.o.g. $r\geqq p^*$ or r, p^* are incompatible in P; in the second case $r\Vdash$ "$\underset{\sim}{T}^*=\underset{\sim}{S}$" so we finish. In the first case clearly as $(q,(s,\underset{\sim}{S}))\in\mathscr{D}$, $q\Vdash$ "$\eta\notin\underset{\sim}{S}$", but $(q,(s,\underset{\sim}{S}))\geqq(p,(t,\underset{\sim}{T}))$ hence $q\Vdash$ "$\eta\notin\underset{\sim}{T}$", hence $p^*_\eta\Vdash$ "$\eta\notin\underset{\sim}{T}'$" (see its definition above). So $r\geqq p^*$, $p^*\geqq q$, $p^*\geqq p^*_\eta$, $q\Vdash_P$ "$\eta\notin\underset{\sim}{S}$", $p^*_\eta\Vdash_P$ "$\eta\notin\underset{\sim}{T}'$" and $p^*\Vdash_P$ "$\underset{\sim}{T}^*=\underset{\sim}{T}'\cup\underset{\sim}{S}$". Clearly this implies $r\Vdash_P$ "$\eta\notin\underset{\sim}{T}^*$". So we have proved $(p^*,(s,\underset{\sim}{T}^*))\in\mathscr{D}$.

The problematic point is checking $(p^*,(s,\underset{\sim}{T}^*))E_n(q,(s,\underset{\sim}{S}))$. Condition (β) is trivial, (γ) holds by Subclaim 7.8(2), and the choice of k, p^* and conditions (α), (ε) too are clear: So we are left with condition (δ); so suppose there is $r_0\in A_m$, $r_0\geqq p^*$, q and $\eta\in{}^{n>}2$. If there is $r\in A_m$ and $r\Vdash_P$ "$\eta\notin\underset{\sim}{T}^*$", then by $\underset{\sim}{T}^*$'s definition, $r\Vdash_P$ "$\eta\notin\underset{\sim}{S}$".

Now suppose there is $r\in A_m$, and $r\Vdash_P$ "$\eta\notin\underset{\sim}{S}$". As A_m is directed, $r_0\in A_m$, w.l.o.g. $r\geqq r_0\geqq q$ hence $r\Vdash_P$ "$\underset{\sim}{T}\subseteq\underset{\sim}{S}$" (as $q\Vdash$ "$(t,\underset{\sim}{T})\leqq(s,\underset{\sim}{S})$").

So clearly $r\Vdash_P$ "$\eta\notin\underset{\sim}{T}$". Hence (as $(p',(t,\underset{\sim}{T}'))E_k(p,(t,\underset{\sim}{T}))$, $k>k_2$ and there is $r'\in A_m$, $r'\geqq p$ (use $r_0\geqq p^*\geqq p$), there is $r'\in A_m$, $r'\Vdash_P$ "$\eta\notin\underset{\sim}{T}'$", w.l.o.g. $r'=r$. As $r\Vdash_P$ "$\eta\notin\underset{\sim}{T}'$" and $r\Vdash_P$ "$\eta\notin\underset{\sim}{S}$", clearly $r\Vdash_P$ "$\eta\notin\underset{\sim}{T}^*$", so we finish the proof of condition (δ) (for m), hence of condition (d) of Definition 7.2(1).

7.9. Definition.

(1) A sweetness model is $(P,\mathscr{D},E_n)_{n<\omega}$, P a forcing notion such that $\mathscr{D}$, E_n $(n<\omega)$ exemplify its sweetness. We allow one to write BA(P) instead of P.

(2) For sweetness models $(P^l,\mathscr{D}^l,E^l_n)_{n<\omega}$ for $l=1,2$, $(P^1,\mathscr{D}^1,E^1_n)_{n<\omega}<(P^2,\mathscr{D}^2,E^2_n)_{n<\omega}$ *if*
   - (a) $P^1<P^2$, $\mathscr{D}^1\subseteq\mathscr{D}^2$, E^1_n is E^2_n restricted to $\mathscr{D}^1$,
   - (b) $p\in\mathscr{D}^1$, $n<\omega$ implies $p/E^2_n\subseteq P^1$,
   - (c) $p\leqq q$, $p\in\mathscr{D}^2$, $q\in\mathscr{D}^1$ implies $p\in\mathscr{D}^1$.

7.10. CLAIM.

(1) $<$ is a quasi-order.

(2) If $(P^k, \mathscr{D}^k, E^k_n)_{n<\omega} < (P^{k+1}, \mathscr{D}^{k+1}, E^{k+1}_n)_{n<\omega}$ then $(\bigcup_{k<\omega} P^k, \bigcup_{k<\omega} \mathscr{D}^k, \bigcup_{k<\omega} E^k_n)$ is a sweetness model and $(P^k, \bigcup_{k<\omega} \mathscr{D}^k, E^k_n)_{n<\omega} < (\bigcup_{k<\omega} P^k, \bigcup_{k<\omega} \mathscr{D}^k, \bigcup_{k<\omega} E^k_n)$.

(3) If $(P, \mathscr{D}, E_n)_{n<\omega}$ is a sweetness model, $P < Q$, P dense in Q, then $(Q, \mathscr{D}, E_n)_{n<\omega}$ is a sweetness model $> (P, \mathscr{D}, E_n)_{n<\omega}$.

PROOF. Easy.

7.11. THE COMPOSITION CLAIM. If $(P^0, \mathscr{D}^0, E^0_n)_{n<\omega}$ is a sweetness model then there is a sweetness model $(P^1, \mathscr{D}^1, E^1_n)_{n<\omega} > (P^0, \mathscr{D}^0, E^0_n)$, $P^1 = P^0 * \underset{\sim}{\mathrm{UM}}$ (when we identify $p \in P_0$ with $p = (p, \varnothing) \in P^0 * \underset{\sim}{\mathrm{UM}}$; remember $\varnothing \in \underset{\sim}{\mathrm{UM}}$ is a minimal element).

PROOF.

As in Composition Lemma 7.6, we can define $\mathscr{D}$, E_n $(n < \omega)$ exemplifying the sweetness of $P^0 * \underset{\sim}{\mathrm{UM}}$ and w.l.o.g. $\mathscr{D} \cap P^0 = \varnothing$. We let

(α) $\mathscr{D}^1 = \mathscr{D}^0 \cup \mathscr{D}$,

(β) $r_1 E^1_n r_2$ if and only if $r_1, r_2 \in \mathscr{D}_1$, $r_1 E_n r_2$ *or* $r_1, r_2 \in \mathscr{D}^0$, $r_1 E^0_n r_2$.

Clearly P^1 is a quasi-order, and $\mathscr{D}$ is a dense subset of it. Hence P^1, $\mathscr{D}$ are equivalent, hence P^1, $P^0 * \underset{\sim}{\mathrm{UM}}$ are equivalent. It is also clear that $P^0 < P^1$, P^1/P is equivalent to $\mathrm{UM}^{V^{P^0}}$. Also conditions (a), (b), (c) of Definition 7.9(2) are obvious. But we have to check $(P^1, \mathscr{D}^1, E^1_n)$ is a sweetness model. In Definition 7.2(1) only part (d) is problematic. Let $r_1 < r_2$, $n < \omega$; if $r_1, r_2 \in \mathscr{D}$ or $r_1, r_2 \in \mathscr{D}^0$ then we use the sweetness of $(P^0, \mathscr{D}^0, E^0_n)_{n<\omega}$, $(P^1, \mathscr{D}^1, E^1_n)_{n<\omega}$ respectively. Otherwise necessarily $r_1 \in P^0$, $r_2 = (p, q) \in \mathscr{D}^0$, hence r_1, p are in $\mathscr{D}^0$. Now by Subclaim 7.8(1), for some n_1, $p' \in p/E_{n_1}$, $p' \geqq p$ implies $(p', q) E^1_n (p, q)$, and by the sweetness of $(P^0, \mathscr{D}^0, E^0_n)$ for some k, $r'_1 \in r_1/E^0_k$ implies r'_1, p have a common bound in $p/E^0_{n_1}$, as $r_1 \leqq p$. So clearly k is as required.

7.12. THE AMALGAMATION CLAIM. Suppose $(P_1, \mathscr{D}_1, E^1_n)_{n<\omega}$, $(P_2, \mathscr{D}_2, E^2_n)_{n<\omega}$ are sweetness models, $P_0 < \mathrm{BA}(P_1)$, $P_0 < \mathrm{BA}(P_2)$. Suppose that we identify $p \in P_1$ with $(p, \phi) \in P_1 *_{P_0} P_2$. Then there are $\mathscr{D}$, E_n $(n < \omega)$ such that $(P_1 *_{P_0} P_2, \mathscr{D}, E_n)_{n<\omega}$ is a sweetness model $> (P_1, \mathscr{D}_1, E_n)_{n<\omega}$.

PROOF. By Lemma 7.5 there are D^*, E^*_n exemplifying the sweetness of $P_1 *_{P_0} P_2$, as defined there. By Lemma 7.3(2), w.l.o.g. $\mathscr{D}^* \cap P_1 = \varnothing$. Let

(α) $\mathscr{D} = \mathscr{D}^* \cup \mathscr{D}^*_1$,

(β) $r_1 E_n r_2$ if and only if $r_1, r_2 \in \mathscr{D}^*$, $r_1 E^*_n r_2$ or $r_1, r_2 \in \mathscr{D}_1$, $r_1 E^1_n r_2$.

We first check that $(P_1 *_{P_0} P_2, \mathscr{D}, E_n)_{n<\omega}$ is a sweetness model as in Claim 7.11, and then it is even easier to check

$$(P_1, \mathscr{D}_1, E^1_n)_{n<\omega} < (P_1 \underset{P_0}{*} P_2, \mathscr{D}_1, E_n)_{n<\omega}.$$

7.13. CLAIM. If $(P, \mathscr{D}, E_n)_{n<\omega}$ is a sweetness model, f an isomorphism from B_1 onto B_2 where B_1, B_2 are complete Boolean subalgebras of $\mathrm{BA}(P)$, then there is a sweetness model $(Q, \mathscr{D}', E'_n)_{n<\omega} > (P, \mathscr{D}, E_n)_{n<\omega}$ and complete Boolean subalgebras B'_1, B'_2 of $\mathrm{BA}(Q)$, and an isomorphism f' from B'_1 onto B'_2, such that: $B_1 < B'_1$, $B_2 < B'_2$, $f \subseteq f'$ and

(1) $B'_1 = \mathrm{BA}(P)$ (as $P < Q$, $\mathrm{BA}(P)$ is a complete Boolean subalgebra of $\mathrm{BA}(Q)$), or even
(2) $B'_1 = B'_2 = \mathrm{BA}(Q)$.

PROOF.
(1) This is a restatement of 7.12 (more exactly, a particular case of it).
(2) Define by induction on l, $(P^l, \mathscr{D}^l, E^l_n)_{n<\omega}$, an increasing sequence of sweetness models, and complete Boolean subalgebras B^l_1, B^l_2 of $\mathrm{BA}(P^l)$, and an isomorphism f^l from B^l_1 onto B^l_2, such that $B^0_1 = B_1$, $B^0_2 = B_2$, $f^0 = f$, $(P^0, \mathscr{D}^0, E^0_n)_{n<\omega}$, and $B^{2l+1}_1 = \mathrm{BA}(P^{2l})$, $B^{2l+2}_2 = \mathrm{BA}(P^{2l+1})$. The induction step is by 7.13(1), and at last let

$$P' = \bigcup_{l<\omega} P^l, \qquad \mathscr{D}' = \bigcup_{l<\omega} \mathscr{D}^l, \qquad E_n = \bigcup_{l<\omega} E^l_n$$

and so $\bigcup_{l<\omega} f^l$ is an automorphism of $\bigcup_l \mathrm{BA}(P^l)$, hence there is a unique extension f' to an automorphism of its completion which is $\mathrm{BA}(\bigcup_{l<\omega} P^l) = \mathrm{BA}(P')$. By Claim 7.10, $(P', \mathscr{D}', E'_n)_{n<\omega}$ is a sweetness model $> (P^l, \mathscr{D}^l, E^l_n)_{n<\omega}$ for each l, hence (for $l = 0$) $> (P, \mathscr{D}, E_n)_{n<\omega}$. Of course $f' \geqq f^l \geqq f^0 = f$, so we finish.

7.14. MAIN LEMMA. *Assume* CH *holds. Then there is an increasing continuous sequence of sweetness models* $(P^\alpha, \mathscr{D}^\alpha, E^\alpha_n)_{n<\omega}$ *for* $\alpha < \omega_1$ *such that letting* $P = \bigcup_{\alpha<\omega_1}$, *then*

(*) (a) $\mathrm{BA}(P)$ *satisfies the countable chain condition and* $\mathrm{BA}(P) = \bigcup_{\alpha<\omega_1} \mathrm{BA}(P_\alpha)$,
(b) *for* every *countably generated, complete Boolean subalgebras* B_1, B_2 *of* $\mathrm{BA}(P)$ *and any isomorphism f from* B_1 *onto* B_2, f can be *extended to an automorphism of* $\mathrm{BA}(P)$.
(c) *for every complete, countably generated Boolean subalgebra* B_1, B_2 *of* B, $B_1 \subseteq B_2$, *there is an automorphism* f *of* B, $f \restriction B_1 =$ *the identity, and* B_2,

$f(B_2)$ are freely amalgamated over B_1, i.e., we can embed $B_2 *_{B_1} f(B_2)$ into B by a function which is the identity over B_2,

(d) *for arbitrarily large α*

$$\mathrm{BA}(P_{\alpha+1})/\mathrm{BA}(P_\alpha) = \mathrm{UM}^{P_\alpha}.$$

PROOF. Trivial by 7.10, 7.11, 7.12, 7.13.

7.15. CLAIM. Forcing with UM makes the union of all "old" closed nowhere-dense subsets of ${}^{\omega}2$ ("old" means with a definition in the ground universe, but allowing members to be new reals) into a set of the first category.

PROOF. Trivial (see Definition 7.7).

7.16. MAIN THEOREM. (1) *For every universe V of set theory satisfying the continuum hypothesis, there is a generic extension V^P in which every set of reals, defined (in V^P) by a first-order formula with a real and ordinal parameter, has the Baire property.*

PROOF. By Solovay [7] from the Main Lemma 7.14 and Claim 7.15.

Again by Solovay [7]:

7.17. CONCLUSION. The following theories are equiconsistent.

(1) ZFC,

(2) ZFC + "every set of reals definable by a first-order formula with ordinal and a real parameter has the Baire property",

(3) ZF + DC + "every set of reals has the Baire property".

REMARK. The proof of 7.17 is, in essence, like this: (1) $\Rightarrow$ (2) by the forcing P from 7.14; (1) $\Rightarrow$ (3) as we can take the subuniverse of V^P consisting of sets hereditarily definable from a real and ordinal parameter; now (2) $\Rightarrow$ (1) trivially and (3) $\Rightarrow$ (1) by Godel's work on L.

CONCLUDING REMARKS.

(1) The proof will be much shorter if we were able to waive Definition 7.2(1)(d), but this causes difficulty in Claim 7.4.

(2) It is not clear whether sweetness is preserved by composition. It would be true if we were to waive Definition 7.2(1) part (d), but even so our proof (of Composition Lemma 7.6) works for some class of forcing notion, but as we have no other example in mind we have not carried this out.

(3) Even the product of $\aleph_1$ Cohen forcing (with finite support) is not sweet (because of Definition 7.2(1) part (b)).

(4) In fact if $P_n < P_{n+1}$, each P_n is sweet, then $\bigcup_{n<\omega} P_n$ is sweet.

CLAIM. If $P < Q$, Q sweet, then P is equivalent to a sweet forcing.

PROOF. Define on Q an order $<^*$:

$$q_1 \leqq^* q_2 \text{ iff (a) there is } p \in P, p \leqq q_2, p \Vdash_P \text{“} q_1 \in Q/P \text{”}$$
$$\text{or (b) } q_1 \leqq q_2.$$

It is easy to see that $\leqq^*$ is transitive, $Q^* = (\{q : q \in Q\}, \leqq^*)$ is equivalent to P (and P is a dense subset) and the $\mathscr{D}$, E_n exemplifying the sweetness of Q exemplify the sweetness of Q^*.

§8. The uniformization property for the Baire category

The uniformization property is a strengthening of “every definable set of reals (with real parameters only) has the Baire property” and Solovay proves that it holds in the model he uses.

The construction in the last section suffices to prove that it is consistent (if ZFC is consistent).

8.1. THEOREM. *Suppose for simplicity $V = L$. Then some inner model of a forcing extension V^P of V (with P of power $\aleph_1$, satisfying the ccc) satisfies $ZF + DC$ and*

(a) *Every set of reals has the Baire property.*

(b) *Suppose for $x \in \mathbf{R}$, A_x is a non-empty set of reals and $\langle A_x : x \in \mathbf{R}\rangle$ is a set. Then there is a function h,* Dom $h = \mathbf{R}$, *$h(x) \in A_x$ except for a first-category set of reals.*

REMARKS. (1) We know V^P itself satisfies (a) and (b) for sets definable with real and ordinal parameters.

(2) Woodin has showed that if $V \models \mathrm{ZF} + \mathrm{DC} + $(b), then V can be elementarily embedded into some forcing extension of it. (Force by Borel non-first-category sets, then define a filter D on R, and embed V into $V^{\mathbf{R}}/D$, $x \to \langle x : x \in R\rangle/\mathscr{D}$.)

PROOF. The proof is like that of Solovay [7], except that we use Lemma 8.2 (see below) at one crucial point. We define a finite support iteration $\langle P_i, \underset{\sim}{Q}_j : i \leqq \omega_1, j < \omega_1\rangle$, where P_i is sweet, as in 7.14, and let $P = P_{\omega_1}$.

We shall eventually use the class of sets in V^P which are hereditarily definable by a real and a member of V. So we can work in V^P; note that for every $\beta < \omega$, $\{r : r \text{ a real in } V^P, r \text{ not Cohen generic over } V^{P_\beta}\}$ is (in V^P) of the first category (as for some $\gamma > \beta$, $Q_\gamma = \underset{\sim}{\mathrm{UM}}$).

Now suppose $\varphi(y, x, a)$ is a formula defining $y \in A_x$, a a real (we suppress the parameters from V). So for some $\alpha < \omega_1$, $a \in V^{P_\alpha}$.

Now suppose $x \in V^P$ is Cohen generic over V^{P_α}, so for some y and β, $\alpha < \beta < \omega_1$, x, $y \in V^{P_\beta}$ and $V^P \models \varphi[y, x, a]$, w.l.o.g. let $\underset{\sim}{x}$, $\underset{\sim}{y}$ be the P_β-name $\Vdash_{P_\beta}$ "$\varphi(y, \underset{\sim}{x}, a)$ and $\underset{\sim}{x}$ is Cohen generic over V^{P_α}".

We use Lemma 8.2 with P_α for P_0, P_β for P_2, $\underset{\sim}{x}$ for $\{\eta : (p, \eta) \in \underset{\sim}{G}_{P_1}\}$. If for some $\gamma < \omega_1$, $(\underset{\sim}{P}_\gamma, \underset{\sim}{D}_\gamma, \underset{\sim}{E}^\gamma_n)_{n<\omega}$ is like $(P, \underset{\sim}{D}, E_n)$ in Lemma 8.2, then we can finish as in Solovay [7]. But we could have replaced α by any α', $\alpha < \alpha' < \omega_1$. So there is no problem to guarantee this in constructing the iteration.

8.2. Lemma *Suppose*

(a) $(P_l, \mathscr{D}_l, E^l_n)_{n<\omega}$ *are sweetness models for* $l = 0, 1, 2$, *and* $(P_0, \mathscr{D}_0, E^0_n)_{n<\omega} < (P_2, \mathscr{D}_2, E^2_n)_{n<\omega}$ *and* $P_0 < P_1 < P_2$.

(b) $P_1 = \{(p, \eta) : p \in P_0, \eta \in {}^{\omega>}\omega\}$ *with* η, $(\emptyset, \eta)$ *and* p, $(p, \emptyset)$ *identified and* $(p, q) \leqq (p', q')$ *if and only if* $p \leqq p'$ (*in* P_0) *and* $\eta \trianglelefteq \eta'$ (*i.e.*, η *is an initial segment of* η') (*so* P_1/P_0 *is Cohen forcing*).

Then *there is a sweetness model* $(P, \mathscr{D}, E_n)_{n<\omega}$ *such that*:

(1) $(P_0, \mathscr{D}_0, E^0_n)_{n<\omega} < (P, \mathscr{D}, E_n)_{n<\omega}$.

(2) *For any* $G_0 \subseteq P_0$ *generic over* V, *in* $V[G_0]$ *the following holds*: $\Vdash_{P/P_0}$ "*there is a function* H *from* P_2/P_0 *to subsets of* ${}^{\omega>}\omega$, *such that*: *if* $G \subseteq {}^{\omega>}\omega$ *is generic over* $V[G_0][\underset{\sim}{G}_{P/P_0}]$ *then* $\{p \in P_2/P_0 : H(p) \cap G \neq \emptyset\}$ *is a directed subset of* P_2/P_0, *generic over* $V[G_0]$, *and including* G".

Proof. Let P be the set of functions f such that:

(α) The domain $\operatorname{Dom} f$ of f is a finite subset of ${}^{\omega>}\omega$ closed under initial segments, which is not empty.

(β) $f(\eta) \in \mathscr{D}_2$ for $\eta \in \operatorname{Dom} f$.

(γ) If $\eta \triangleleft \nu$ are both in $\operatorname{Dom} f$, then $f(\eta) \leqq f(\nu)$ (in P_2).

(δ) There are $n < \omega$, and $r \in P_0$ such that for any $p'_\eta \in f(\eta)/E^2_n$ (for $\eta \in \operatorname{Dom} f$) there is $r' \in r/E^2_0 = r/E^0_0$ such that

$$r' \Vdash_{P_0} [p'_\eta, \eta \text{ belong to } P_2/P_0, \text{moreover } \eta \Vdash_{P_1/P_0} \text{“} p'_\eta \in (P_2/P_0)/(P_1/P_0)\text{”}].$$

Let, for $f \in P$, $n(f)$ be the minimal n for which (δ) holds. The order on P is defined by:

$$f_1 \leqq f_2 \quad \text{iff } \operatorname{Dom} f_1 \subseteq \operatorname{Dom} f_2 \quad \text{and for } \eta \in \operatorname{Dom} f_1, \quad f_1(\eta) \leqq f_2(\eta).$$

Now we shall define the sweetness model. First, let $\mathscr{D} = P$. Second, we have to define its equivalence relations. As a preliminary step, we define for every $f \in \mathscr{D}$ and $n < \omega$ a function k_f from $\operatorname{Dom} f$ to ω. The definition is carried by downward induction on the length of $\eta \in \operatorname{Dom} f$ (this is possible as $\operatorname{Dom} f$ is finite). So we should have written $k_{f,n}(\eta)$.

Let $k_f(\eta)$ be the minimal k which is $\geqq n$, $\geqq n(f)$ and for every $\eta\,^\wedge\langle i\rangle \in \operatorname{Dom} f$, for every $q' \in f(\eta)/E^2_k$ there is $r \in f(\eta\,^\wedge\langle i\rangle)/E^2_{k_f(\eta^\wedge\langle i\rangle)}$, $q' \leqq r$. There is such a k as $\eta\,^\wedge\langle i\rangle \in \operatorname{Dom} f$ for finitely many i's only, and condition (δ) of the definition of the sweetness model.

So now at last we come to defining the E_n's:

fE_nh iff f, h have the same domain, the functions k_f, k_h are equal, and

$$f(\eta)E_{k_f(\eta)}h(\eta) \quad \text{when } \eta \in \operatorname{Dom} f.$$

8.3. FACT. In the definition of E_n we can drop "$k_{f_1} = k_{f_2}$". Moreover *if* $f \in \mathscr{D}$, h a monotonic function from $\operatorname{Dom} f$ to $\mathscr{D}_2$, and $f(\eta)E_{k_f}h(\eta)$ for every $\eta \in \operatorname{Dom} H$, *then* $h \in \mathscr{D}$, fE_nh.

PROOF. Clearly it suffices to prove the "moreover". Now by $n(f)$'s definition there is r such that:

$$\text{for every } p'_\eta \in f(\eta)/E^2_n\ (\eta \in \operatorname{Dom} f) \text{ there is } r' \in r/E^2_0 = r/E^0_n, \text{ s.t.}$$
$$r' \Vdash_{P_0} \text{“}p'_\eta,\ \eta \text{ belong to } P_2/P_0, \text{ moreover } \eta \Vdash_{P_1/P_0} p'_\eta \in (P_2/P_0)/(P_1/P_0)\text{”}.$$

But $p'_\eta \in f(\eta)/E^n_2$ iff $p'_\eta \in h(\eta)/E^2_n$. So h satisfies condition (δ) of the definition of P. Now (γ) was assumed (h monotonic) as well as (β) and (α) (as $f \in P$). As Range $h \subseteq \mathscr{D}_2$, $h \in \mathscr{D}$.

Now r, $n(f)$ witness $n(h) \leqq n(f)$, and the inverse inequality is proved similarly (as $f(\eta)E^2_{n(f)}h(\eta)$ implies that $h(\eta)/E^2_m = f(\eta)/E^2_m$ for $m \leqq n(f)$). So $n(f) = n(h)$. Now the equality $k_f(\eta) = k_h(\eta)$ is proven by downward induction on η. So we have proved Fact 8.3.

Now we can easily prove that $(P, \mathscr{D}, E_n)_{n<\omega}$ is a sweetness model.

Now note that $P_0 \not\subseteq P$, whereas we want $(P_0, \mathscr{D}_0, E_n)_{n<\omega} < (P, \mathscr{D}, E_n)$. So let

$P' = P_0 \cup P$ (assuming they are disjoint),
$p \leqq q$ (in P') *iff* $p, q \in P_0$, $P_0 \vDash p \leqq q$ or $p, q \in P$, $P \vDash p \leqq q$
or $p \in P_0$, $q \in P$, $\operatorname{Dom} q \neq \emptyset$
and $p \leqq q(\eta)$ for every $\eta \in \operatorname{Dom} q$,

$$\mathscr{D}' = \mathscr{D}_0 \cup \mathscr{D},$$

$$E'_n = E^0_n \cup E_n.$$

Now the rest is trivial.

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